SOME TOPICS IN PROBABILITY MEASURES ON SEMIGROUPS

by

LUK MEE-LIN

A thesis

Submitted to

the Graduate School of

The Chinese University of Hong Kong

(Division of Mathematics)

In Partial Fulfilment

of the Requirement for the Degree of

Master of Philosophy (M. Phil.)

May 1989

ABSTRACT

Given a compact semigroup S, the set P(S) of probability measures (i.e. normalized non-negative regular Borel measures) on S forms a compact affine semigroup under convolution and with the relative topology induced by the weak*-topology for $C(S)^*$, the dual of the Banach space of all continuous complex valued functions on S. Probability measures have been studied by various mathematicians in the past thirty years. Among these studies the case of idempotent and primitive idempotent measures has received special consideration and led to some significant results.

We consider algebraic semigroups in Chapter 1 and topological semigroups in Chapter 2. In these chapters we gather some facts and tools for later use in Chapter 3.

The first topic in our work is about elements of finite order which is a generalized concept of idempotents. We extend the natural partial order in idempotents to elements of finite order in Chapter 1. In P(S), with the help of supports of measures, properties of primitive idempotent and central primitive idempotent measures are enriched. Next we take up again the topic about averaged product sequences and ultra-convex sets in an affine semigroup considered in Chapter 2. In P(S) further results are obtained. Then we discuss nilpotent elements when P(S) has zero. These so called nilpotent measures can be characterized in terms of their supports. Finally we investigate properties of bonds in P(S) and we hope their properties can be helpful to the study of probability measures on semigroups.

ACKNOWLEDGEMENT

I would like to express my deep gratitude to Dr. H.L. Chow for his advice and encouragement during the preparation of this thesis.

ABSTRACT

Given a compact semigroup S, the set P(S) of probability measures (i.e. normalized non-negative regular Borel measures) on S forms a compact affine semigroup under convolution and with the relative topology induced by the weak*-topology for $C(S)^*$, the dual of the Banach space of all continuous complex valued functions on S. Probability measures have been studied by various mathematicians in the past thirty years. Among these studies the case of idempotent and primitive idempotent measures has received special consideration and led to some significant results.

We consider algebraic semigroups in Chapter 1 and topological semigroups in Chapter 2. In these chapters we gather some facts and tools for later use in Chapter 3.

The first topic in our work is about elements of finite order which is a generalized concept of idempotents. We extend the natural partial order in idempotents to elements of finite order in Chapter 1. In P(S), with the help of supports of measures, properties of primitive idempotent and central primitive idempotent measures are enriched. Next we take up again the topic about averaged product sequences and ultra-convex sets in an affine semigroup considered in Chapter 2. In P(S) further results are obtained. Then we discuss nilpotent elements when P(S) has zero. These so called nilpotent measures can be characterized in terms of their supports. Finally we investigate properties of bonds in P(S) and we hope their properties can be helpful to the study of probability measures on semigroups.

CONTENTS

List of symbols

Suppose S is an algebraic semigroup or a topological semigroup, a is an element of S and A is a subset of S. Suppose, when S is compact, μ is an element of $P(S)$ and Ω is a subset of $P(S)$.

1. $C(S)$: the Banach space of all continuous complex valued functions on S with the supremum norm.
2. $C(S)^*$: the dual space of $C(S)$.
3. $E(S)$: the set of idempotents in S.
4. $K(S)$ or K : the minimal ideal of S.
5. $P(S)$: the semigroup of probability measures on S.
6. $\Gamma(A)$: the closed subsemigroup of S generated by A.
7. $K(a)$: the set of cluster points of the sequence (a^n).
8. $E(a)$: the identity of $K(a)$.
9. $L(a)$: the limit of the sequence (a_n) where $a_n = \frac{1}{n}(a + .. + a^n)$.
10. $J_0(A)$: the union of ideals of S contained in A.
11. $\overline{co}(A)$: the closed convex hull of A.
12. $\partial(A)$: the set of extreme points of A.
13. $\leq$: $a \leq b$ if $a = a^n b = ab^n = b^n a = ba^n$.
14. $\leq_1$: $a \leq_1 b$ if $a = a^n b = ba^n$.
15. $\leq_2$: $a \leq_2 b$ if $a = a^n b = b^n a$.
16. $\leq_3$: $a \leq_3 b$ if $a = ab^n = ba^n$.
17. $I_n(S)$ or I_n : $\{x \in S : x^{n+1} = x\}$.

18. Π_n : the set of primitive elements with respect to $\leq$ in $I_n(S)$.

19. Π_n^i : the set of primitive elements with respect to $\leq_i$ in $I_n(S)$.

20. $C\Pi_n$: the set of central elements in Π_n .

21. $C\Pi_n^i$: the set of central elements in Π_n^i .

22. Q_α : $\{x \in S : L(x) = e_\alpha\}$ where e_α is the idempotent corresponding to Q_α.

23. supp Ω : the support of Ω.

24. $\Lambda(\text{supp } \mu)$: $\{\nu \in P(S)$: each neighbourhood of ν meets supp μ^n from some n on$\}$.

25. δ_a : the Dirac measure at a.

26. K_4 : the Klein 4-group.

27. S_3 : the symmetric group on 3 symbols.

28. Z_n : the cyclic group of order n.

Chapter 1 Algebraic semigroups

§1 Basic definitions and properties

In this section we give some basic definitions and properties of algebraic semigroups for the later work.

1.1.1. Definitions.

An algebraic semigroup, or simply a semigroup, is a non-empty set S together with a function $(a,b) \longmapsto ab$ from $S \times S$ into S such that the associative law holds, i.e. $(ab)c = a(bc)$ for all a, b, $c \in S$. The function is usually called the multiplication. By abuse of language, we shall call S itself an algebraic semigroup.

An element $1 \in S$ with $1a = a1 = a$ for all $a \in S$ is uniquely determined, if it exists, and is called the identity of S.

An element $0 \in S$ with $0a = a0 = 0$ for all $a \in S$ is uniquely determined, if it exists, and is called the zero of S.

An element $e \in S$ with $e^2 = e$, if it exists, is called an idempotent.

Let A and B be subsets of S. The product of A and B is the set $AB = \{ab : a \in A, b \in B\}$. A non-empty subset T of S is a subsemigroup (left ideal, right ideal, ideal) of S if $TT \subset T$ (respectively, $ST \subset T$, $TS \subset T$, $ST \cup TS \subset T$).

A minimal ideal (left ideal, right ideal) of S is an ideal (left ideal, right ideal) of S containing no other ideal (left ideal, right ideal) of S properly.

Throughout the remainder of this chapter S is always an algebraic semigroup unless the contrary is explicitly specified.

1.1.2. Facts and examples.

(1) For any two ideals I, J of S, the non-empty set IJ is contained in both I and J. Also $I \cap J$ is an ideal of S. Thus any two ideals must intersect and the intersection is again an ideal. Hence the intersection of all ideals of S, if non-empty, is the smallest ideal of S. Thus, if S has a minimal ideal, then it is unique and is denoted by K(S), or simply K when no confusion can arise.

The minimal ideal of a semigroup may not exist in general. For example, let $S = (0,1)$ with the usual multiplication. Then any ideal of S is of the form of $(0,a)$ or $(0,a]$ where $0 < a < 1$. Thus the empty intersection gives that S has no minimal ideal.

(2) For any two left (right) ideals I, J of S, $I \cap J$ may be empty. If $I \cap J$ is non-empty, then $I \cap J$ is again a left (right) ideal. Hence a semigroup may have no minimal left (right) ideal or may have more than one minimal left (right) ideal. For the latter case, each minimal left (right) ideal is disjoint from others.

For example, let $S = [0,1]$ with the multiplication defined by $ab = b$ for all $a, b \in S$. Then S is a semigroup and each subset of S is a left ideal of S. Thus any singleton subset of S is a minimal left ideal of S. That is, S has infinitely many minimal left ideals.

(3) Ideals of S are not transitive in general, in the sense that suppose J is an ideal of S and I is an ideal of J, but I may not be an ideal of S.

For example, let $S = [0,1]$ with the usual multiplication, $J = [0,\frac{1}{2}]$ and $I = [0,\frac{1}{4}] \cup \{\frac{1}{2}\}$. Then J is an ideal of S and I is an ideal of J. But I is not an ideal of S since $\frac{7}{8} \in S$, $\frac{1}{2} \in I$ and $\frac{7}{8} \times \frac{1}{2} = \frac{7}{16} \notin I$.

1.1.3. Proposition.

Let J be an ideal of S and I an ideal of J. Then :

(1) If $I^2 = I$, then I is an ideal of S.

(2) If J is the minimal ideal of S, then I is an ideal of S. In fact $I = J$.

Proof. (1) $IS = I^2S \subset IJS \subset IJ \subset I$. Similarly $SI \subset I$. Thus I is an ideal of S.

(2) Since $JIJS \subset JIJ$ and $SJIJ \subset JIJ$, JIJ is an ideal of S. Hence $J \subset JIJ$. Since I is an ideal of J, then $JIJ \subset I$. Thus $J \subset JIJ \subset I \subset J$ gives $I = J$ and so I is an ideal of S.

1.1.4. Definitions.

S is called simple if S has no proper ideal. Thus the minimal ideal of a semigroup, if it exists, is a simple semigroup by 1.1.3. A necessary and sufficient condition for S to be simple is that $SxS = S$ for all $x \in S$. [25, P.35]

An element $a \in S$ is called central if $ax = xa$ for all $x \in S$. An element $a \in S$ is said to be normal if $aS = Sa$.

1.1.5. Proposition.

A normal idempotent element e of S is central.

Proof. Let x be an arbitrary element of S. Since e is normal, there exist y, z $\in$ S such that ex = ye and xe = ez. Since (ex)e = (ye)e = ye = ex and e(xe) = e(ez) = ez = xe, we have ex = xe. Thus e is central.

1.1.6. Proposition. [15, P.22-23]

Let e $\in$ S be an idempotent. Then e is contained in a unique maximal subgroup of S and any two distinct maximal subgroups are disjoint.

1.1.7. Definitions.

An ideal P of S is said to be prime if AB $\subset$ P implies A $\subset$ P or B $\subset$ P, where A and B are ideals of S.

An ideal P of S is said to be completely prime if ab $\in$ P implies a $\in$ P or b $\in$ P, where a, b $\in$ S.

An ideal P of S is said to be semiprime if for some n $\in \mathbb{N}$, $a^n \in$ P implies a $\in$ P.

1.1.8. Proposition.

Let P be a completely prime ideal of S. Then P is prime and semiprime.

Proof. Let A and B be ideals of S and AB $\subset$ P. Suppose A $\not\subset$ P and B $\not\subset$ P. Then there exist a $\in$ A \ P and b $\in$ B \ P. Since ab $\in$ AB $\subset$ P and P is completely prime, it follows that a $\in$ P or b $\in$ P, a contradiction. Thus $A \subset P$ or $B \subset P$, i.e. P is prime.

Suppose $a^n \in P$ for some $n \in \mathbb{N}$. Since P is completely prime, we have $a \in P$ or $a^{n-1} \in P$. If $a^{n-1} \in P$, then we have $a \in P$ or $a^{n-2} \in P$. Repeat the argument if necessary; then after a finite number of steps, we have $a \in P$, i.e. P is semiprime.

1.1.9. Examples.

The converse of the above proposition is false.

(1) Let $S = \{a, b, c\}$ with the multiplication table as follow :

	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

The associativity can be checked by Light's test given in [15, P.7-8]. Then S is a semigroup and $P = \{a\}$ is a semiprime ideal. But P is not prime since $A = \{a, b\}$, $B = \{a, c\}$ are ideals of S and $AB = \{a\} \subset P$ with $A \not\subset P$ and $B \not\subset P$. Hence P is not completely prime too.

(2) Let $S = \{a, b, c, d, e\}$ with the multiplication table as follow:

	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	e
c	a	a	c	d	a
d	a	d	a	a	c
e	a	a	e	b	a

Then S is a semigroup . It is easy to check that $\{a\}$ and S are the only ideals of S. Clearly $\{a\}$ is prime but not semiprime since $d^2 = a$ and so not completely prime.

In conclusion, we have the following figure concerning about relations among completely prime ideals, prime ideals and semiprime ideals.

$$\begin{tikzcd}
& \text{completely prime ideals} \arrow[dl, shift right] \arrow[dr, shift left] & \\
\text{prime ideals} \arrow[ur, shift right, "/" marking] \arrow[rr, shift left, "/" marking] & & \text{semiprime ideals} \arrow[ul, shift left, "/" marking] \arrow[ll, shift left, "/" marking]
\end{tikzcd}$$

However, if S is commutative, it is easy to see that an ideal of S being prime is equivalent to be completely prime.

1.1.10. <u>Definitions.</u>

An ideal M of S is said to be maximal if it is proper and is not properly contained in any proper ideal of S.

A maximal ideal M of S is said to be trivial if M is of the form of $S \setminus \{a\}$ for some $a \in S \setminus S^2$.

1.1.11. <u>Facts and examples.</u>

(1) A prime ideal may not be maximal.

For example, let $T = [0,1]$ with the usual multiplication. Let $S = T \cup \{a\}$ and define a multiplication $*$ on S by

$$x * y = \begin{cases} xy & \text{if } x,\ y \in T \\ 0 & \text{if } x \in T,\ y = a \quad \text{or} \quad x = a,\ y \in T \\ a & \text{if } x = y = a \end{cases}$$

Then S is a semigroup and $P = \{0, a\}$ is clearly a prime ideal of S. But $Q = [0,\frac{1}{2}] \cup \{a\}$ is an ideal of S containing P. Thus P is not maximal.

(2) A maximal ideal need not be prime.

For example, let $S = [1, \frac{1}{2}]$ with the usual multiplication. Then $M = S \setminus \{\frac{1}{2}\}$ is a maximal ideal but not a prime ideal since $S^2 \subset M$ while $S \not\subset M$.

(3) A prime ideal is never a trivial maximal ideal.

Proof. Suppose not, if prime ideal $P = S \setminus \{a\}$ with $a \in S \setminus S^2$ then $S^2 \subset P$. Since P is prime, we have $S \subset P$, a contradiction.

1.1.12. <u>Proposition.</u>

A maximal ideal of S is either trivial or prime.

<u>Proof.</u> Let M be a maximal ideal of S. Assume M is not trivial, i.e. M is not of the form of $S \setminus \{a\}$ for some $a \in S \setminus S^2$. Then $Q = S \setminus M$ contains either an idempotent or an element $b = xy$ where $x, y \in S$. In fact $x, y \in Q$; then $b \in Q \cap Q^2$. In any case, we have $Q \cap Q^2 \neq \emptyset$. Let A and B be ideals of S with $AB \subset M$. Suppose $A \not\subset M$ and $B \not\subset M$. Since M is a maximal ideal, we have $M \cup A = S = M \cup B$. Thus $Q \subset A$ and $Q \subset B$, then $Q^2 \subset AB \subset M$. Since $Q^2 \cap Q \neq \emptyset$, then $Q \cap M \neq \emptyset$, a contradiction. Hence it follows that M is prime.

1.1.13. <u>Corollary.</u>

(1) Every non-trivial maximal ideal is prime.

(2) If $S^2 = S$, then every maximal ideal is prime.

(3) If $S \setminus M$ contains an idempotent or more than one element, then the maximal ideal M is prime.

<u>Proof.</u> Clear.

§2 Elements of finite order in semigroups

In this section we study extensions of the partial order for idempotents of semigroups to elements of finite order. We shall take up elements of finite order again in Chapter 3 in which semigroups of probability measures on a compact semigroup are considered.

1.2.1. Definitions.

Let S be a semigroup. $a \in S$ is called an element of finite order if $a^{n+1} = a$ for some positive integer n.

Denote the set $\{a \in S : a^{n+1} = a\}$ by $I_n(S)$. When no confusion can arise, we write I_n for $I_n(S)$. It is obvious that $I_n \supset I_1$ and $I_n \cap I_{n+1} = I_1$. Note that I_1 is the set of idempotents of S. The natural ordering on I_1 is defined as follow : for $e, f \in I_1$, we say $e \leq f$ if $e = ef = fe$.

H.L. Chow has extended the order from I_1 to I_n. For $a, b \in I_n$, define $a \leq b$ by $a = a^n b = ab^n = b^n a = ba^n$. It is shown in [4] that the relation $\leq$ defined in I_n is a partial order extended from I_1.

An element $a \in I_n$ is said to be primitive in I_n if a is the zero element 0 of S or a minimal non-zero element in I_n, i.e. for any $b \in I_n$ and $b \leq a$ we have $b = 0$ or $b = a$.

We denote by $\Pi_n(S)$, or simply Π_n, the set of primitive elements in I_n. The set of central elements in Π_n, i.e. $\{x \in \Pi_n : xy = yx \text{ for all } y \in S\}$, is denoted by $C\Pi_n$.

Some elementary properties concerned about Π_n are given in the following proposition. The proof is straight forward and therefore is omitted.

1.2.2. <u>Proposition.</u>

(1) $\Pi_n \supset \Pi_1$.

(2) $\Pi_n \cap I_1 = \Pi_1$, i.e. an idempotent is primitive in I_n if and only if it is primitive in I_1.

(3) $\Pi_n \cap \Pi_{n+1} = \Pi_1$.

The next proposition is a sufficient condition for an element of finite order to be primitive.

1.2.3. <u>Theorem.</u>

Suppose $a \in I_n$. If $a^n \in \Pi_1$, then $a \in \Pi_n$.

<u>Proof.</u> Let $b \in I_n$ and $b \leq a$, i.e. $b = b^n a = ba^n = a^n b = ab^n$. Then $b^n = b^{n-1}b = b^{n-1}ba^n = b^n a^n$ and $b^n = bb^{n-1} = a^n bb^{n-1} = a^n b^n$. Since $a^n, b^n \in I_1$, we have $b^n \leq a^n$. As $a^n \in \Pi_1$, then $b^n = 0$ or $b^n = a^n$. If $b^n = 0$, then $b = b^n b = 0b = 0$. If $b^n = a^n$ then $b = b^n a = a^n a = a$. Therefore a is primitive in I_n.

1.2.4. <u>Example.</u>

Hence the converse of 1.2.3 does not hold in general.

Let $S = \{a, b, c, d\}$ with the multiplication given by

	a	b	c	d
a	a	a	a	a
b	d	c	b	a
c	a	b	c	d
d	d	d	d	d

Then S is a semigroup (checked by Light's test in [15, P.7-8]). It is easily verified that $b \in \Pi_2$, but $b^2 = c \notin \Pi_1$ since $a = ac = ca$, i.e. $a \leq c$.

However, we shall see in Chapter 3 (3.2.11) that it is true in the semigroup P(S) of probability measures on a compact semigroup S.

For a partial converse of 1.2.3, we have the following theorem.

1.2.5. Theorem.

Suppose $a \in \Pi_n$. Any one of the following conditions implies $a^n \in \Pi_1$.

(1) a is central.

(2) Every idempotent is central.

(3) The minimal ideal K exists and is a non-trivial group.

(4) S is an inverse semigroup (i.e. for any $x \in S$, there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$) and a union of groups.

(5) S is a left group (or right group), i.e. S contains an idempotent element and $S = Sx$ (or $S = xS$) for all $x \in S$.

Proof. (1) Let $b \in I_1$ and $b \leq a^n$, i.e. $b = ba^n = a^n b$. Since a is central, we have $(ab)^{n+1} = a^{n+1}b^{n+1} = ab$ and thus $ab \in I_n$. It is easy to see that $ab = (ab)^n a = (ab)a^n = a^n(ab) = a(ab)^n$, i.e. $ab \leq a$. Since $a \in \Pi_n$, we have $ab = 0$ or $ab = a$. If $ab = 0$, then $b = a^{n-1}(ab) = 0$. If $ab = a$, then $b = a^n b = a^{n-1}(ab) = a^n$. Hence a^n is primitive in I_1.

(2) Same as (1) since b is central.

(3) Let e be the identity of the group K. First, we show that e is the smallest element in I_1. Let $f \in I_1$; then ef, $fe \in K$. We have $ef = (ef)e = e(fe) = fe$ and $(ef)^2 = efef = eeff = ef$. Thus $ef \in I_1$. Similarly $fe \in I_1$. Since e, ef and fe are idempotents in the group K, we have $e = ef = fe$. Thus $e \leq f$, i.e. e is the smallest idempotent.

Next we assert that $a^n \in \Pi_1$. Since $e \in K$, we have ae, $ea \in K$. Thus $ae = e(ae) = (ea)e = ea$. Then $(ae)^{n+1} = a^{n+1}e^{n+1} = ae$, i.e. $ae \in I_n$. Since $(ea)^n a = e^n a^{n+1} = ea = a(ea)^n$ and $a^n(ea) = ea^{n+1} = ea = (ea)a^n$, we have $ea \leq a$. Since a is primitive and S has no zero because of K being non-trivial, we have $ea = a$ and so $a \in K$. As $a^{n+1} = a^n a = a$, we have $a^n = e \in \Pi_1$, i.e. a^n is primitive in I_1.

(4) It follows from [15, P.127] that every idempotent element in S is a central element. By (2) above the result follows.

(5) Let $b \in I_1$ and $b \leq a^n$, i.e. $b = ba^n = a^n b$. Since S is a left group, then by [15, P.38] S is right cancellative. Thus $bb = a^n b$ implies $b = a^n$. Hence a^n is primitive in I_1.

1.2.6. Theorem.

If $a \in C\Pi_n$, then $a^n \in C\Pi_1$.

Proof. As $a \in \Pi_n$ and a is central, by the above theorem we have $a^n \in \Pi_1$. Since a is central, then a^n is central. Hence $a^n \in C\Pi_1$.

1.2.7. Corollary.

If $a \in C\Pi_n$, then $a^i \in C\Pi_n$ for $i = 1,..,n$.

Proof. Since $a \in C\Pi_n$, then by the above theorem $a^n \in C\Pi_1$. Clearly $a^i \in I_n$ and $(a^i)^n = a^n \in C\Pi_1$. By 1.2.3 we have $a^i \in \Pi_n$. Hence $a^i \in C\Pi_n$.

1.2.8. Example.

The converse of 1.2.6 may fail to hold.

Consider S_3 the symmetric group on 3 symbols, i.e. $S_3 = \langle x, y \mid x^3 = y^3 = e, xy = yx^2\rangle$ where e is the identity of S_3. Then $y \in I_2$ and $y^2 = e \in C\Pi_1$. But $y \notin C\Pi_2$ since y is not central.

However, the converse of 1.2.6 is true in P(S) (3.2.12).

1.2.9. Theorem.

Suppose $a \in I_n$. a is normal if and only if a^n is central.

Proof. Suppose a is normal and so is a^n. Since a^n is idempotent, by 1.1.5. a^n is central. Conversely, if a^n is central, then for any $b \in S$ we have $ab = aa^nb = aba^n = (aba^{n-1})a$. Thus $ab \in Sa$ and so $aS \subset Sa$. Similarly $Sa \subset aS$. Thus $aS = Sa$, i.e. a is normal.

1.2.10. Remark.

A normal element of finite order need not be central, Referring to 1.2.8. $y \in I_2$ is normal but y is not central. However, a normal primitive element of finite order must be central in $P(S)$ (3.2.13).

In general, the sets I_n and Π_n are not semigroups. An example will be given in Chapter 3 (3.2.17). But $C\Pi_n$ is a semigroup and its structure is considered in the following results.

1.2.11. Lemma.

(1) If S has zero element 0, then the product of two different elements in $C\Pi_1$ is 0.

(2) If S has no zero element, then $C\Pi_1$ is empty or a singleton $\{e\}$. In the latter case, e is the smallest element in I_1 and so $\Pi_1 = \{e\}$.

Proof. (1) Let $e, f \in C\Pi_1$. Then $(ef)^2 = efef = eeff = ef$; thus $ef \in I_1$. Also $e(ef) = eef = ef$ and $(ef)e = eef = ef$, we have $ef \leq e$ and so $ef = 0$ or e. Thus, if $e \neq f$, we have $ef = 0$.

(2) Suppose $C\Pi_1 \neq \emptyset$. Let $e, f \in C\Pi_1$; then $ef \in I_1$ and $ef \leq e$. Since e is primitive and S has no zero element, we have $ef = e$. Similarly $ef = f$ and so $e = f$. Thus $C\Pi_1$ is a singleton, say $\{e\}$. Let g be any element in I_1. Then $eg = ge \in I_1$ and $eg \leq e$. Thus $eg = e$ and so $e \leq g$. Hence e is the smallest element in I_1.

Remark. We shall see in Chapter 3 (3.2.14) that if S is a semigroup of probability measures having no zero element, then $C\Pi_1 = \emptyset$. But if S is a group with identity e, then $C\Pi_1 = \{e\}$, i.e. $C\Pi_1 \neq \emptyset$.

1.2.12. Theorem.

If $C\Pi_n \neq \emptyset$, then $C\Pi_n$ is a semigroup which is a mutually disjoint union of groups.

Proof. By 1.2.11 $C\Pi_1$ is a semigroup. Let $a, b \in C\Pi_n$. By 1.2.6 we have $a^n, b^n \in C\Pi_1$. Thus $(ab)^n = a^n b^n \in C\Pi_1$. As $(ab)^{n+1} = a^{n+1}b^{n+1} = ab$, i.e. $ab \in I_n$, by 1.2.3 we have $ab \in \Pi_n$. Since ab is central, then $ab \in C\Pi_n$. Hence $C\Pi_n$ is a semigroup.

For each $a \in C\Pi_n$, a^n is an idempotent in the semigroup $C\Pi_n$ and so generates a maximal group $H(a^n)$ in $C\Pi_n$. (Note $H(a^n)$ may not be maximal in S.) Since $aa^{n-1} = a^n$ and by 1.2.7 $a^{n-1} \in C\Pi_n$, we have $a \in H(a^n)$. Hence $C\Pi_n = \bigcup_{a\in C\Pi_n} H(a^n)$ which is a mutually disjoint union of groups by 1.1.6.

1.2.13. Example.

For the semigroup of probability measures on a compact semigroup, we shall see that $H(a^n) = \{a, a^2, \ldots, a^n\}$ (elements may not be distinct) in the previous theorem. However this is not true for general semigroups.

Let S be the Klein 4-group K_4 with the zero adjoined, i.e. $S = \langle x, y, z \mid x^2 = y^2 = z^2 = e,\ xy = z\rangle \cup \{0\}$. Then $C\Pi_2 = S = H(0) \cup H(x^2)$. But $H(x^2) = K_4$ which is not a cyclic group.

1.2.14. Definitions.

Now we look a bit closer to the order defined on I_n. We have said that for a, b $\in I_n$, $a \leq b$ if and only if $a = a^n b = ab^n = b^n a = ba^n$. In fact, we observe that this relation is equivalent to $\leq_1$ which is defined as follow : a, b $\in I_n$, $a \leq_1 b$ if $a = a^n b = ba^n$. Since if $a = a^n b = ba^n$, i.e. $a \leq_1 b$, we have $ab^n = a^n bb^n = a^n b = a$ and $b^n a = b^n ba^n = ba^n = a$, i.e. $a \leq b$. Hence $a \leq_1 b$ implies $a \leq b$. Conversely, it is clear that $a \leq b$ implies $a \leq_1 b$. Hence orders $\leq_1$ and $\leq$ are equivalent.

We consider the five other combinations of $a^n b$, ab^n, $b^n a$ and ba^n. Define $a \leq_2 b$ if and only if $a = a^n b = b^n a$.

$a \leq_3 b$ if and only if $a = ab^n = ba^n$.

$a \leq_4 b$ if and only if $a = a^n b = ab^n$.

$a \leq_5 b$ if and only if $a = b^n a = ba^n$.

$a \leq_6 b$ if and only if $a = ab^n = b^n a$.

We find that the first three relations are partial orders in I_n while the latter three are not.

1.2.15. Examples.

The following examples show that anti-symmetric property is not satisfied in the relations $\leq_4$, $\leq_5$ and $\leq_6$. Hence they are not partial orders.

(1) Let S = {a, b, c} with the multiplication table as follow:

	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Then S is a semigroup and $I_1 = \{a, b, c\}$. Since $ab = a$ and $ba = b$, we have $a \leq_4 b$ and $b \leq_4 a$, but $a \neq b$.

(2) Let $S = \{a, b, c\}$ with the multiplication table as follow:

	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

Similar to (1), $a \leq_5 b$ and $b \leq_5 a$, but $a \neq b$.

(3) Let $S = \{a, b, c\}$ with the multiplication table defined by

	a	b	c
a	a	a	c
b	a	a	c
c	c	c	a

Then S is a semigroup and $I_2 = \{a, c\}$. Since $ac^2 = c^2a = a$ and $ca^2 = a^2c = c$, we have $a \leq_6 c$ and $c \leq_6 a$, but $a \neq c$.

Since $\leq$ is a partial order, it follows that $\leq_1$ is a partial order. We show that $\leq_2$ is a partial order. By symmetry $\leq_3$ is also a partial order.

1.2.16. Proposition.

$\leq_2$ is a partial order in I_n.

Proof. We have defined that $a \leq_2 b$ if and only if $a = a^n b = b^n a$. Since $a \in I$, then $a = a^{n+1} = a^n a$, i.e. $a \leq a$. Thus reflexivity of $\leq_2$ is clear.

Secondly we check anti-symmetricity. Let $a, b \in I_n$ with

$a \leq_2 b$ and $b \leq_2 a$, i.e. $a = a^n b = b^n a$ and $b = b^n a = a^n b$. Thus $a = a^n b = b$, i.e. $a = b$.

Finally we verify the transitive property. Let $a, b, c \in I_n$ with $a \leq_2 b$ and $b \leq_2 c$. i.e. $a = a^n b = b^n a$ and $b = b^n c = c^n b$. We assert that $a \leq_2 c$, i.e. $a = a^n c = c^n a$. Now $a = a^n b = a^n b^n c$ $= a^{n-1} a\, b^n c = a^{n-1} a^n b b^n c = a^{n-1} a^n bc = a^{n-1} ac = a^n c$. Also $a = b^n a$ $= bb^{n-1} a = c^n bb^{n-1} a = c^n b^n a = c^n a$. Thus $a \leq_2 c$. Hence $\leq_2$ is a partial order.

1.2.17. Examples.

Note that $\leq_1$, $\leq_2$ and $\leq_3$ are distinct orders. In fact, $a \leq_1 b$ implies $a \leq_2 b$ and $a \leq_3 b$, but not vice versa.

(1) Let $S = \{a, b, c\ d\}$ with the multiplication table as follow:

	a	b	c	d
a	a	a	a	a
b	d	c	b	a
c	a	b	c	d
d	d	d	d	d

Then S is a semigroup and $I_2 = \{a, b, c, d\}$. Since $a = a^2 b$ $= b^2 a$, i.e. $a \leq_2 b$, but $ba^2 = d \neq a$. Thus $a \not\leq_1 b$ and $a \not\leq_3 b$ as well.

(2) Let $S = \{a, b, c, d\}$ with the multiplication table as follow:

	a	b	c	d
a	a	d	a	d
b	a	c	b	d
c	a	b	c	d
d	a	a	d	d

Similar to (1), we have $a \leq_3 b$, but $a \not\leq_1 b$ and $a \not\leq_2 b$.

The following are some equivalence conditions for $\leq_1$, $\leq_2$ and $\leq_3$.

1.2.18. Proposition.

For $a, b \in I_n$, the following statements are equivalent.

(1) $a \leq_1 b$, i.e. $a = a^n b = ba^n$.

(2) $a = a^n b = ba^n = ab^n = b^n a$.

(3) $a^2 = ab = ba$.

Proof. We have seen that (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (3) : $a^2 = aa^n b = ab$ and $a^2 = ba^n a = ba$.

(3) $\Rightarrow$ (1) : $a^n b = a^{n-1}ab = a^{n-1}a^2 = a^{n+1} = a$ and

$ba^n = baa^{n-1} = a^2 a^{n-1} = a^{n+1} = a$.

1.2.19. Proposition.

For $a, b \in I_n$, the following conditions are equivalent.

(1) $a \leq_2 b$, i.e. $a = a^n b = b^n a$.

(2) $a = a^n b = b^n a = ab^n$.

(3) $a^n = a^n b^n = b^n a^n$ and $a^2 = ab$.

Proof. (1) $\Rightarrow$ (2) : $ab^n = a^n bb^n = a^n b = a$.

(2) ⇒ (3) : $a^n = a^{n-1}a = a^{n-1}ab^n = a^nb^n$ and

$a^n = aa^{n-1} = b^naa^{n-1} = b^na^n$.

Also $a^2 = a(a^nb) = a^{n+1}b = ab$.

(3) ⇒ (1) : $a = a^{n+1} = a^{n-1}a^2 = a^{n-1}ab = a^nb$ and

$a = a^{n+1} = a^na = b^na^na = b^na$.

1.2.20. Proposition.

For $a, b \in I_n$, the following conditions are equivalent.

(1) $a \le_3 b$, i.e. $a = ab^n = ba^n$.

(2) $a = ab^n = ba^n = b^na$.

(3) $a^n = a^nb^n = b^na^n$ and $a^2 = ba$.

Proof. Similar to 1.2.19.

1.2.21. Remark.

For $a, b \in I_n$, any one of $a \le_1 b$, $a \le_2 b$ and $a \le_3 b$ implies $a^n = a^nb^n = b^na^n$. Hence $\le_1$, $\le_2$ and $\le_3$ are extensions of the partial order in I_1 which is defined by $e \le f$ if and only if $e = ef = fe$, where $e, f \in I_1$.

1.2.22. Notation.

Let Π_n^i denote the set of all primitive elements in I_n with respect to $\le_i$ and $C\Pi_n^i$ the set of central elements in Π_n^i, where i = 1, 2, 3.

Note that Π_n^1 is the set Π_n and thus has the properties discussed before. Since $\le_1$, $\le_2$ and $\le_3$ are identical in I_1, then $\Pi_1^1 = \Pi_1^2 = \Pi_1^3$. Thus they are simply denoted by Π_1.

1.2.23. Proposition.

$$C\Pi_n^1 = C\Pi_n^2 = C\Pi_n^3.$$

Proof. Let $a \in C\Pi_n^1$ and $b \in I_n$ with $b \leq_2 a$. Then $b = b^n a = ab^n$ since a is central. Thus $b \leq_1 a$. Since a is primitive with respect to $\leq_1$, we have $b = 0$ or a. Thus $a \in C\Pi_n^2$ and so $C\Pi_n^1 \subset C\Pi_n^2$. Conversely, let $a \in C\Pi_n^2$ and $b \in I_n$ with $b \leq_1 a$. Then $a^n b = a^n b^n b = a^{n-1} b^2 = \dots = ab^n = b$ and so $b = b^n a = a^n b$, i.e. $b \leq_2 a$. Since a is primitive with respect to $\leq_2$, we have $b = 0$ or a. Thus $a \in C\Pi_n^1$. Therefore $C\Pi_n^2 \subset C\Pi_n^1$ and so $C\Pi_n^1 = C\Pi_n^2$. Similarly $C\Pi_n^1 = C\Pi_n^3$.

1.2.24. Proposition.

$$\Pi_1 \subset \Pi_n^2 \text{ and } \Pi_1 \subset \Pi_n^3.$$

Proof. Let $a \in \Pi_1$. Then $a \in I_1$ and thus $a \in I_n$. Let $b \in I_n$ and $b \leq_2 a$, i.e. $b = b^n a = a^n b$. Therefore $b = b^n a = b^n aa = ba$, $b = a^n b = ab$ and $b^2 = a^n b b^n a = a^n ba = ba = b$. Thus $b \in I_1$ and $b \leq a$. Since $a \in \Pi_1$, we have $b = 0$ or a. Thus a is primitive in I_n, i.e. $\Pi_1 \subset \Pi_n^2$. The relation $\Pi_1 \subset \Pi_n^3$ can be shown similarly.

1.2.25. Proposition.

$$\Pi_n^2 \subset \Pi_n^1 \text{ and } \Pi_n^3 \subset \Pi_n^1.$$

Proof. Let $a \in \Pi_n^2$ and $b \in I_n$ with $b \leq_1 a$. Thus $b \leq_2 a$. Since $a \in \Pi_n^2$, thus $b = 0$ or a, i.e. $a \in \Pi_n^1$. Similarly we have $\Pi_n^3 \subset \Pi_n^1$.

1.2.26. Examples.

From 1.2.24 and 1.2.25 we have $\Pi_1 \subset \Pi_n^2$, Π_n^3 and Π_n^2 , $\Pi_n^3 \subset \Pi_n^1$. Therefore we have $\Pi_1 \subset \Pi_n^2 \cap \Pi_n^3$ and $\Pi_n^2 \cup \Pi_n^3 \subset \Pi_n^1$. The following examples show that these two inclusions may be strict.

(1) Let S = {a, b, c} with the multiplication table as follow :

	a	b	c
a	a	a	c
b	a	a	c
c	c	c	a

Then S is a semigroup, $I_1 = \{a\} = \Pi_1$ and $I_2 = \{a, c\}$. Since S is commutative, we have $\Pi_2^1 = \Pi_2^2 = \Pi_2^3$. As $a^2c = ac = c \neq a$, we have $a \not\leq_1 c$. Thus $\Pi_2^2 = \Pi_2^3 = \{a, c\}$. Hence the inclusion $\Pi_1 \subset \Pi_2^2 \cap \Pi_2^3$ is strict.

(2) Let S = {a, b, c, d, e} with the multiplication table as follow :

a	a	b	c	d	e	f	g
a	d	e	c	a	b	f	g
b	b	b	g	b	b	g	g
c	f	g	c	c	g	f	g
d	a	b	c	d	e	f	g
e	e	e	g	e	e	g	g
f	c	g	c	f	g	f	g
g	g	g	g	g	g	g	g

Then S is a semigroup, $I_1 = \{b, c, d, e, f, g\}$,
$I_2 = \{a, b, c, d, e, f, g\}$ and $\Pi_1 = \{b, c, e, f, g\}$.
From $b^2a = ba = b$ and $a^2b = db = b$, we have $b \leq_2 a$, thus $a \notin \Pi_n^2$.

From $ca^2 = cd = c$ and $ac^2 = ac = c$, we have $c \leq_3 a$, thus $a \notin \Pi^3_n$.

Since $ab^2 = ab = e \neq b$, i.e. $b \not\leq_1 a$,

$c^2a = ca = f \neq c$, i.e. $c \not\leq_1 a$,

$d^2a = da = a \neq d$, i.e. $d \not\leq_1 a$,

$ae^2 = ae = b \neq e$, i.e. $e \not\leq_1 a$,

$f^2a = fa = c \neq f$, i.e. $f \not\leq_1 a$, thus $a \in \Pi^1_2$.

Thus $\Pi^1_2 = \{a, b, c, e, f, g\}$, $\Pi^2_2 = \{b, c, e, f, g\}$

and $\Pi^3_2 = \{b, c, e, f, g\}$.

Hence the inclusion $\Pi^2_2 \cup \Pi^3_2 \subset \Pi^1_2$ is strict.

1.2.26. Remark.

Although the sets Π^1_n, Π^2_n and Π^3_n may be distinct, Π^2_n and Π^3_n have the same properties of Π^1_n shown before. Proofs are parallel to those given in 1.2.2 to 1.2.12. However in Chapter 3 (3.2.10) we shall see that sets Π^1_n, Π^2_n and Π^3_n coincide in P(S).

Chapter 2 Topological semigroups

§1 Basic facts

In this section we give some basic definitions and properties of topological semigroups, especially those of compact semigroups, for the later work.

2.1.1. Definition.

An algebraic semigroup S is said to be a topological semigroup if S is a Hausdorff topological space such that the multiplication, i.e. the mapping $(x,y) \longmapsto xy$ from $S \times S$ to S is continuous. In addition, if S is a compact space, then S is called a compact semigroup.

Suppose S is a topological semigroup and T is an algebraic subsemigroup of S. Then T is also a topological semigroup under the relative topology.

2.1.2. Examples.

The following are some examples of topological semigroups.

(1) $S = [0,1]$ with the usual multiplication and the usual topology.

(2) $S = [0,1]$ with the usual topology and the multiplication defined by $xy = \min \{x, y\}$ for all $x, y \in S$.

(3) $S = [0,1]$ with the usual topology and the multiplication defined by $xy = x$ for all $x, y \in S$.

(4) $S = \{a, b\}$ with the discrete topology and the multiplication defined by

	a	b
a	a	a
b	a	b

2.1.3. Lemma.

Let S be a topological semigroup and A, B subsets of S. Then $\overline{AB} = \overline{\bar{A}\bar{B}}$ where the bars denote closure. Hence $\overline{AB} = \bar{A}\bar{B}$ if S is compact.

Proof. Since $A \subset \bar{A}$ and $B \subset \bar{B}$, then $AB \subset \bar{A}\bar{B}$. Thus $\overline{AB} \subset \overline{\bar{A}\bar{B}}$. Conversely, it suffices to show $\bar{A}\bar{B} \subset \overline{AB}$. Let $x \in \bar{A}\bar{B}$, i.e. $x = yz$ for some $y \in \bar{A}$ and $z \in \bar{B}$. Then there exist nets $(y_\alpha)_{\alpha\in\Lambda_1}$ in A and $(z_\beta)_{\beta\in\Lambda_2}$ in B such that $y_\alpha \to y$ and $z_\beta \to z$. Fix β in Λ_2, we have $y_\alpha z_\beta \to yz_\beta$ by the continuity of the multiplication. Since $y_\alpha z_\beta \in AB$ for all $\alpha \in \Lambda_1$ and $\beta \in \Lambda_2$, we have $yz_\beta \in \overline{AB}$. Now vary β and take limit $yz_\beta \to yz \in \overline{AB}$, i.e. $x \in \overline{AB}$. Thus $\bar{A}\bar{B} \subset \overline{AB}$.

If S is compact, then $\bar{A}$ and $\bar{B}$ are compact. By Tychonoff's theorem, $\bar{A} \times \bar{B}$ is compact. Since the multiplication is continuous, then $\bar{A}\bar{B}$ is compact. Thus $\bar{A}\bar{B}$ is closed. Therefore $\overline{\bar{A}\bar{B}} = \bar{A}\bar{B}$. Hence $\overline{AB} = \bar{A}\bar{B}$.

By the nets argument as above and the continuity of the multiplication, it is easy to have the following proposition.

2.1.4. Proposition.

Let S be a topological semigroup and $A \subset S$. Taking closure of A preserves the following properties :

(1) A being a subsemigroup of S, i.e. $\overline{A}$ is still a subsemigroup of S.

(2) A being an ideal of S.

(3) A being a simple subsemigroup of S, if S is compact.

2.1.5. Proposition. [25, P.22,32]

If S is a compact semigroup, then there exists a minimal ideal of S and S contains at least one idempotent.

In the rest of this section, S *is always a compact semigroup unless the contrary is explicitly specified.*

2.1.6. Notation.

Let $a \in S$. Denote by $\Gamma(a)$ the closed subsemigroup generated by a, i.e. $\Gamma(a) = \overline{\cup \{a^n : n \in \mathbb{N}\}}$ and by $K(a)$ the set of cluster points of the sequence $(a^n)_{n \in \mathbb{N}}$.

2.1.7. Proposition. [25, P.109]

Let $a \in S$. Then :

(1) $K(a)$ is a commutative subgroup of $\Gamma(a)$ and $\Gamma(a)$ contains exactly one idempotent which is the identity of $K(a)$, denoted by $E(a)$.

(2) $K(a)$ is the minimal ideal of $\Gamma(a)$ and $K(a) = H_a(E(a))$ where $H_a(E(a))$ is the maximal subgroup generated by $E(a)$ in $\Gamma(a)$.

2.1.8. Notation.

Suppose S is an algebraic semigroup and $A \subseteq S$. We define $J_\circ(A)$ to be the empty set if A contains no ideal of S, otherwise $J_\circ(A)$ is the union of all ideals contained in A. Thus if $J_\circ(A) \neq \emptyset$, then $J_\circ(A)$ is the largest ideal of S contained in A.

2.1.9. Proposition. [26]

(1) Let A be an open subset of S. Then $J_\circ(A)$ is open.

(2) Let e be an idempotent. Then $J_\circ(S\backslash\{e\})$, if it is non-empty, is an open prime ideal of S.

(3) Let P be a proper open prime ideal of S. Then $P = J_\circ(S\backslash\{e\})$ for some idempotent e.

2.1.10. Proposition. [25, P.28]

Any proper ideal of S is contained in a maximal ideal of S and each maximal ideal is open.

2.1.11. Proposition.

The following sets concerning about finite order of elements of S are closed.

(1) $I_n = \{x \in S : x^{n+1} = x\}$.

(2) $N_n = \{x \in I_n : x \text{ is normal}\}$.

(3) $C\Pi_n = \{x \in \Pi_n : x \text{ is central}\}$.

Proof. (1) Let $x \in \overline{I_n}$. Then there exists a net (x_α) in I_n such that $x_\alpha \to x$. Thus $x_\alpha^n \to x^n$. Since $x_\alpha^n = x_\alpha$ for all α, it follows that $x^{n+1} = x$, i.e. $x \in I_n$. Hence I_n is closed.

(2) Let $x \in \overline{N_n}$. Then there exists a net (x_α) in I_n such that $x_\alpha \to x$. Since I_n is closed, then $x \in I_n$. In view of 1.2.9 x_α^n is central for each α, it gives that x^n is central and so x is normal. Thus $x \in N_n$, i.e. N_n is closed.

(3) We first assert that $C\Pi_1$ is closed. Let $x \in \overline{C\Pi_1}$. There exists a net (x_α) in $C\Pi_1$ such that $x_\alpha \to x$. It is clear that $x \in I_n$ and x is central. Let $y \in I_1$ with $y \leq x$, i.e. $y = xy$. Let $y_\alpha = x_\alpha y$. Since $x_\alpha y \to xy = y$, we have $y_\alpha \to y$. Since $y_\alpha^2 = (x_\alpha y)(x_\alpha y) = x_\alpha x_\alpha yy = x_\alpha y = y_\alpha$, i.e. $y_\alpha \in I_1$. As $x_\alpha y_\alpha = x_\alpha x_\alpha y = x_\alpha y = y_\alpha$, we have $y_\alpha \leq x_\alpha$ and so $y_\alpha = 0$ or x_α. It follows that the only possible cluster points of the nets (y_α) are 0 or x. Consequently $y = 0$ or $y = x$. Hence $x \in C\Pi_1$ and so $C\Pi_1$ is closed.

To show that $C\Pi_n$ is closed we take $x \in \overline{C\Pi_n}$. There exists a net (x_α) in $C\Pi_n$ such that $x_\alpha \to x$. Thus $x \in I_n$ and x is central. By virtue of 1.2.6 $x^n \in C\Pi_1$. That $C\Pi_1$ is closed gives $x^n \in C\Pi_1$. Thus $x^n \in \Pi_1$ and so $x \in \Pi_n$ by 1.2.3. Hence $x \in C\Pi_n$, completing the proof.

§2 Affine semigroups

In this section we consider a special kind of compact semigroup, namely compact affine semigroups. Properties of limits of averaged product sequences and ultra-convex sets in such semigroups are investigated

2.2.1. Definitions.

Let X and Y be Hausdorff topological vector spaces over the real field. Suppose S and T are convex subsets of X and Y respectively. A mapping $\phi : S \to T$ is said to be affine if ϕ preserves convex combinations, i.e. $\phi(tx + (1-t)y) = t\phi(x) + (1-t)\phi(y)$ for $x, y \in S$ and $0 \leq t \leq 1$.

If $\phi : S \to T$ is affine, it is easily checked by induction that $\phi(\sum_{i=1}^{n} t_i x_i) = \sum_{i=1}^{n} t_i \phi(x_i)$ whenever $x_i \in S$, $t_i \geq 0$ for $i = 1,..,n$ and $\sum_{i=1}^{n} t_i = 1$.

A semigroup S is said to be affine if it is a convex subset of a Hausdorff topological vector space over the real field such that $x(ty + (1-t)z) = txy + (1-t)xz$ and $(ty + (1-t)z)x = tyx + (1-t)zx$ for $x, y, z \in S$ and $0 \leq t \leq 1$, i.e. the multiplication in S is separately affine.

If , in addition, the affine semigroup S is also a topological semigroup (relative to the topology inherited from X), then S is called an affine topological semigroup. In particular, S is a compact affine semigroup if S is compact.

An element e of a convex set S of a topological vector space is called an extreme point of S if $e = tx + (1-t)y$ with $x, y \in S$ and $0 < t < 1$ implies $e = x = y$.

By Krein-Milman theorem [20, P.440-441], the set $\partial(S)$ of extreme points of a compact convex subset S of a locally convex space X is non-empty and S is equal to $\overline{co}(\partial(S))$ which is the closed convex hull of $\partial(S)$, i.e. the smallest closed convex set containing $\partial(S)$.

In the sequel, we shall need the following lemma which is a simple consequence of [16, Theorem].

2.2.2. Lemma.

Suppose the compact affine semigroups S is in a locally convex topological vector space. Then an idempotent e of S is extreme on the set eSe, i.e. if $e = ta + (1-t)b$ for $a, b \in eSe$ and $0 < t < 1$, then $a = e = b$.

2.2.3. Examples.

I. The following are some examples of compact affine semigroups.

(1) [0,1] with the usual multiplication and the usual topology.

(2) [0,1] with the usual topology and the multiplication defined by $xy = x$ for all $x, y \in [0,1]$.

(3) The set P(S) of probability measures on a compact semigroup S, under the convolution and the weak*-topology. (P(S) will be discussed in Chapter 3)

II. A convex semigroup may not be affine.

Let S = [0,1] with the usual topology and the multiplication defined by $a * b = \min\{a, b\}$ for all $a, b \in [0,1]$. Then S is a convex semigroup but not affine since $\frac{1}{4} * (\frac{1}{2} \times 0 + \frac{1}{2} \times 1) = \frac{1}{4} * \frac{1}{2} = \frac{1}{4}$ but $\frac{1}{2} \times (\frac{1}{4} * 0) + \frac{1}{2} \times (\frac{1}{4} * 1) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$.

In the rest of this chapter, we shall always assume that S is a compact affine semigroup which lies in a locally convex space if no other information is given.

2.2.4. Definition and remarks.

Given $a \in S$. Let $a_n = \frac{1}{n}(a + a^2 + \dots + a^n)$. The sequence $(a_n)_{n\in\mathbb{N}}$ is called an averaged product sequence in S.

We see in [7] that the sequence (a_n) converges to a point in S denoted by $L(a)$. Moreover $L(a)$ is an idempotent with $L(a)a = aL(a) = L(a)$. In fact $L(a)$ is the zero of $\overline{co}(\Gamma(a))$.

Given $e_\alpha \in E(S)$ which is the set of all idempotents in S. Let $Q_\alpha = \{x \in S : L(x) = e_\alpha\}$. It is clear that the set $\{Q_\alpha : e_\alpha \in E(S)\}$ gives a partition on S.

According to [7], we have $e_\alpha \le e_\beta$ if and only if $e_\beta \in \overline{Q}_\alpha$, where $e_\alpha, e_\beta \in E(S)$.

2.2.5. Proposition.

For any positive integer p, $L(a) = a_p L(a^p) = L(a^p)a_p$.

Proof. Observing that $a^p a_{np} = a_p (a^p)_n$, let $n \to \infty$ and obtain $a^p L(a) = a_p L(a^p)$. Since $L(a) = a^p L(a)$, we have $L(a) = a_p L(a^p)$. Similarly $L(a) = L(a^p)a_p$, giving the result.

2.2.6. Corollary.

If $a = a^{p+1}$, then $L(a) = a_p$.

Proof. Since a^p is an idempotent, therefore $a^p = L(a^p)$. Hence $L(a) = a_p a^p = a_p$.

The following propositions (2.2.7, 2.2.8) are given in [7].

2.2.7. Proposition.

For $a \in S$, the following are equivalent:

(1) $L(a) = E(a)$.

(2) The sequence (a^n) is convergent.

(3) $K(\Gamma(a)) = \{E(a)\}$.

2.2.8. Proposition.

For e_α , $e_\beta \in E(S)$. Then :

(1) The set Q_α is convex and its closure $\overline{Q_\alpha} = \{x \in S : xe_\alpha = e_\alpha x = e_\alpha\}$.

(2) $\overline{Q_\alpha} \cap Q_\beta \neq \emptyset$ if and only if $e_\beta \in \overline{Q_\alpha}$.

(3) e_α is maximal if and only if Q_α is closed.

2.2.9. Proposition.

If Q_α is open, then e_α is minimal.

Proof. Suppose $e_\beta \leq e_\alpha$, i.e. $e_\alpha \in \overline{Q_\beta}$. Since $e_\alpha \in Q_\alpha$ which is open. There exists a neighbourhood V of e_α such that $V \subset Q_\alpha$. Since $e_\alpha \in \overline{Q_\beta}$, then $V \cap Q_\beta \neq \emptyset$ and so $Q_\alpha \cap Q_\beta \neq \emptyset$. Thus $Q_\alpha = Q_\beta$, i.e. $e_\alpha = e_\beta$. Therefore e_α is minimal.

2.2.10. Proposition.

Suppose S has finitely many idempotents. If e_α is minimal, then Q_α is open.

Proof. It suffices to show Q_α contains no limit point of $S \setminus Q_\alpha$.

Suppose there is an element x of Q_α which is also a limit point of $S \setminus Q_\alpha$. Then $x \in \overline{S \setminus Q_\alpha} = \overline{\cup Q_i}$ (where $Q_i \neq Q_\alpha$) $= \cup \overline{Q_i}$ since there are only finite number of Q_i. Thus $x \in \overline{Q_i}$ for some i and so $Q_\alpha \cap \overline{Q_i} \neq \emptyset$. By 2.2.8 we get $e_i \leq e_\alpha$. Since e_α is minimal, then $e_i = e_\alpha$ and so $Q_i = Q_\alpha$, a contradiction. Hence Q_α is open.

2.2.11. Remark.

If S has infinitely many idempotents, then e_α being minimal does not imply that Q_α is open. An example will be given in Chapter 3 (3.3.13).

2.2.12. Proposition. [7]

Let $e_\alpha \in E(S)$, the following are equivalent :

(1) e_α is the zero of S.

(2) Q_α is dense in S.

(3) $\overline{Q_\alpha}$ is an ideal of S.

2.2.13. Remark.

Note that if e_α is the zero of S, then Q_α is an ideal of S. In general Q_α need not be a semigroup. An example will be given in Chapter 3 (3.3.13).

2.2.14. Definition.

A non-empty subset Ω of S is called ultra-convex if $ta + (1-t)b \in \Omega$ whenever $a \in \Omega$, $b \in S$ and $0 < t < 1$.

Hence, if Ω is an ultra-convex subset of S, then $\Omega = \{ta + (1-t)b : a \in \Omega, b \in S \text{ and } 0 < t < 1\}$.

2.2.15. Proposition.

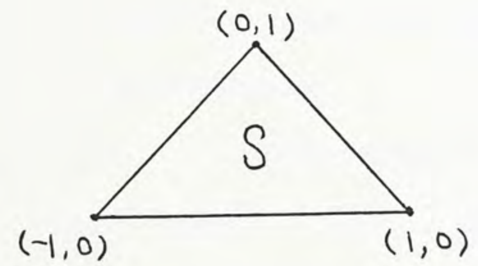
Let Ω be an ultra-convex set of S. Then Ω is convex and dense.

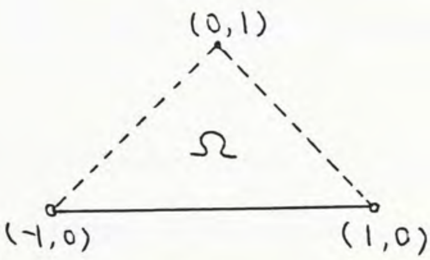
Proof. It is clear that Ω is convex. To see Ω is dense. Let x be any point of S and take $a \in \Omega$. Construct a sequence in Ω by $a_n = \frac{1}{n}a + (1 - \frac{1}{n})x$, where $n \in \mathbb{N}$. Then $a_n \to x$. Hence Ω is dense in S.

2.2.16. Example.

The converse of the above proposition is not true.

Let $S = \{(x,y) : x + y \leq 1,\ y - x < 1 \text{ and } y \geq 0\}$ with the usual coordinate-wise multiplication and the usual topology inherited from $\mathbb{R}^2$. Then S is a compact affine semigroup. The set S is the triangle in $\mathbb{R}^2$ with vertices (0,1), (1,0), and (-1,0). Let $\Omega = \text{Int } S \cup \{(x,0) : -1 < x < 1\} \cup \{(0,1)\}$, i.e.





Then Ω is clearly convex and dense in S but Ω is not ultra-convex since $\frac{1}{2}(1,0) + \frac{1}{2}(0,1) \notin \Omega$.

A trivial ultra-convex set of S is the set S itself. Other examples of ultra-convex sets are given by the following proposition.

2.2.17. Proposition.

Suppose $A \subset S$. Any one of the following conditions implies that A is ultra-convex.

(1) A is an open prime ideal of S.

(2) A is an open semiprime ideal of S.

(3) A is an open completely prime ideal of S.

(4) A is a maximal ideal of S.

(5) Suppose S has a zero and A is the set of all nilpotent elements in S.

Proof. (1) By [7].

(2) By [9].

(3) By 1.1.8 and (1) above.

(4) By 1.1.12, 2.1.10 and (1) above.

(5) By [7].

2.2.18. Examples.

(1) A prime or semiprime ideal of S may not be ultra-convex if it is not open.

Let $S = [0,1]$ with the usual multiplication and the usual topology. Then $P = \{0\}$ is a prime and semiprime ideal of S but not ultra-convex.

(2) In a general compact semigroup, an open prime or semiprime ideal need not be dense (cf. 2.2.15).

Let $S = \{0\} \cup [\frac{1}{2},1]$ with the usual topology and the multiplication defined by $xy = \min \{x, y\}$. Then $\{0\}$ is clearly an open prime and semiprime ideal of S but not dense.

Chapter 3 Probability measures on semigroups

Throughout this chapter, S always denotes a compact semigroup unless the contrary is explicitly specified.

§1 Fundamental concepts

3.1.1. Construction.

Here we outline the construction of a compact affine semigroup, whose elements are probability measures on a given compact semigroup. For further details, we refer to [22].

Let S be a compact semigroup and C(S) the Banach space of all continuous complex valued functions on S with the supremum norm. The dual space $C(S)^*$ is a locally convex space under the weak*-topology (the topology of pointwise convergence), i.e. for any $\mu \in C(S)^*$ and a net (μ_α) in $C(S)^*$, then $\mu_\alpha \to \mu$ if and only if $\mu_\alpha(f) \to \mu(f)$ for all $f \in C(S)$. By Riesz's representation theorem, we can identify $C(S)^*$ with the space of all bounded regular Borel measures on S. We shall use the same letter μ to denote the functional and the measure, writing $\mu(f) = \int_S f(x)d\mu(x)$ where $f \in C(S)$.

Let $P(S) = \{\mu \in C(S)^* : \mu$ is non-negative and $\|\mu\| = \mu(1) = \mu(S) = 1\}$. P(S) is called the set of probability measures on S. It is easy to see that P(S) is convex and by net argument P(S) is closed. In view of Alaoglu theorem, P(S) is a compact convex subset of the locally convex space $C(S)^*$ under the weak*-topology.

For $\mu, \nu \in P(S)$, the convolution of μ and ν is the measure $\mu\nu$ satisfying $\int_S f(z)d\mu\nu(z) = \int_S\int_S f(xy)d\mu(x)d\nu(y)$ for all $f \in C(S)$. By the compactness of S and the continuity of the multiplication in S, it is not hard to see that $\mu\nu \in P(S)$. By Fubini's theorem, the convolution in P(S) is associative and is separately affine by properties of integration. In fact, the convolution is continuous under the weak*-topology. Hence P(S) is a compact affine semigroup under the convolution and the weak*-topology.

3.1.2. Remark.

For $x \in S$, let δ denote the Dirac measure at x, i.e. for any Borel set A in S,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A . \end{cases}$$

Thus $\delta_x(f) = f(x)$ for all $f \in C(S)$. Let $\tilde{S} = \{\delta_x : x \in S\}$. Then, under the induced topology, $\tilde{S}$ is homeomorphic to S. By Krein-Milman theorem [21, P.440-441], we have $P(S) = \overline{co}(\tilde{S})$. By means of net argument we see that a probability measure μ is central in P(S) if $\mu\delta_x = \delta_x\mu$ for all $x \in S$.

3.1.3. Definition.

Let $\mu \in P(S)$. The support of μ denoted by supp μ is the closed set F such that $\mu(F) = 1$ and $\mu(U) > 0$ if U is open and $U \cap F \neq \emptyset$. It can be shown that such closed set exists and is unique [22]. In fact F is the smallest closed subset of S with $\mu(F) = 1$.

The following is a well-known formula for supports of measures in $P(S)$.

3.1.4. Proposition. [22, Lemma 2.1]

Let μ, $\nu \in P(S)$. Then $\operatorname{supp} \mu\nu = (\operatorname{supp} \mu)(\operatorname{supp} \nu)$.

To generalize the concept of supports of elements in $P(S)$ to subsets in $P(S)$, we have the following.

3.1.5. Definition.

Let Λ be a subset of $P(S)$. Then the support of Γ is defined as the set $\operatorname{supp} \Lambda = \overline{\bigcup_{\mu\in\Lambda} \operatorname{supp} \mu}$.

3.1.6. Proposition.

Let Λ, Λ_1 and Λ_2 be subsets of $P(S)$. Then :

(1) $\operatorname{supp}(\Lambda_1\Lambda_2) = (\operatorname{supp} \Lambda_1)(\operatorname{supp} \Lambda_2)$. Hence, if Λ is a subsemigroup (ideal) of $P(S)$, then $\operatorname{supp} \Lambda$ is a subsemigroup (ideal) of S.

(2) $\operatorname{supp} \overline{\Lambda} = \operatorname{supp} \Lambda$.

(3) If Λ is a closed subsemigroup of $P(S)$, then $\operatorname{supp} K(\Lambda) = K(\operatorname{supp} \Lambda)$.

Proof. (1) Clear.

(2) It follows by Urysohn's lemma.

(3) By [24, Theorem 3; 2, Theorem 5].

The following propositions are some useful tools, concerning about idempotent measures, for the later work.

3.1.7. Proposition. [18, Lemma 2; 28, Lemma 3]

Let $\mu \in P(S)$ be an idempotent measure. Then :

(1) supp μ is a compact simple semigroup.

(2) $\mu\nu\mu = \mu$ for $\nu \in P(S)$ with supp $\nu \subset$ supp μ.

3.1.8. Example.

The converse of 3.1.7(1) is false, i.e. $\mu \in P(S)$ and supp μ is a compact simple semigroup but μ may not be idempotent.

Consider Z_2 the cyclic group of order 2, i.e. $\langle x \mid x^2 = e \rangle$. Let $\mu = \frac{1}{3}\delta_e + \frac{2}{3}\delta_x$. Then supp $\mu = \{e, x\} = S$ which is compact and simple, but $\mu^2 = (\frac{1}{3}\delta_e + \frac{2}{3}\delta_a)^2 = \frac{5}{9}\delta_e + \frac{4}{9}\delta_a \neq \mu$.

3.1.9. Fact.

If G is a compact subgroup of S, then m, the (normalized) Haar measure of G, is clearly an idempotent in P(S). The converse is the following proposition.

3.1.10. Proposition. [30, P.925]

Let $\mu \in P(S)$ be an idempotent. If $F =$ supp μ is a group, then μ is the Haar measure of F. Moreover $\mu = \mu\nu = \nu\mu$ for $\nu \in P(S)$ with supp $\nu \subset F$.

3.1.11. Proposition. [12, P.722]

P(S) has the zero ϑ if and only if the minimal ideal K of S is a compact group. In this case, ϑ is the Haar measure of K.

§2 Finite order elements in P(S)

In this section we consider elements of finite order in P(S). With the help of relations between such elements and their supports, we study properties of primitive elements of finite order and answer the questions arose from section 2 of Chapter 1.

3.2.1. Notation.

In this section we always assume that $\mu \in P(S)$ is an element of finite order, i.e. $\mu^{n+1} = \mu$ for some positive integer n. Denote supp μ by F. We write $\Gamma(F)$ for the closed subsemigroup generated by F. Notations introduced in section 2 of Chapter 1 will be used here.

3.2.2. Lemma.

Suppose $\nu \in P(S)$ with supp $\nu \subset F^j$ for some $j = 1,..,n$. Then $\mu^i\nu\mu^k = \mu^{i+j+k}$ for $i,k = 1,..,n$.

Proof. Since supp $\mu^{2n-j}\nu \subset F^{2n-j}F^j = F^{2n} = F^n$, then $\mu^n(\mu^{2n-j}\nu)\mu^n = \mu^n$ by 3.1.7. Thus $\mu^{i+j+k} = \mu^{i+j}\mu^n\mu^k = \mu^{i+j}\mu^n(\mu^{2n-j}\nu)\mu^n\mu^k = \mu^i\nu\mu^k$.

3.2.3. Proposition.

$\Gamma(F) = \bigcup_{i=1}^{n} F^i$ which is a mutually disjoint union of closed sets.

Proof. Since supp $\mu^i = F^i$ and $F^iF^n = F^nF^i = F^i$ for $i = 1,..,n$, it follows that $\Gamma(F) = \bigcup_{i=1}^{n} F^i$. Thus it remains to show $F^i = F^j$ if $F^i \cap F^j \neq \emptyset$. Let $x \in F^i \cap F^j$. In particular since $x \in F^i$, then

$\text{supp}\ \mu^{2n-i}\delta_x \subset F^{2n-i}F^i = F^{2n} = F^n$. By 3.1.7 we have $\mu^n(\mu^{2n-i}\delta_x)\mu^n = \mu^n$. Then $\mu^n\delta_x\mu^n = \mu^{i+n+2n-i}\delta_x\mu^n = \mu^i\mu^n = \mu^i$. Similarly we get $\mu^n\delta_x\mu^n = \mu^j$. Thus $\mu^i = \mu^j$. Hence $F^i = F^j$, completing the proof.

3.2.4. Corollary.

If $\Gamma(F)$ is connected, then μ is an idempotent.

Proof. Obvious.

3.2.5. Theorem.

Let $\mu \in I_n$. Then $\Gamma(F)$ and F^n are compact simple semigroups of S having the same idempotents. Moreover, $\Gamma(F)$ is a group if and only if F^n is.

Proof. Let $\omega = \frac{1}{n}(\mu + .. + \mu^n)$. Then $\omega\mu^i = \mu^i\omega = \omega$ for $i = 1,..,n$. Thus $\omega^2 = \omega$. Also $\text{supp}\ \omega = \text{supp}\ \frac{1}{n}(\mu + .. + \mu^n) = F \cup .. \cup F^n = \Gamma(F)$. In view of 3.1.7 $\Gamma(F)$ and F^n are compact simple subsemigroups of S. Since $F^n \subset \Gamma(F)$, idempotents in F^n are in $\Gamma(F)$. Let e be an idempotent in $\Gamma(F)$. Then $e \in F^i \cap F^{2i}$ for some $i = 1,..,n$. By virtue of 3.2.3 $F^i = F^{2i}$. Thus $F^i = F^{ni} = F^n$, i.e. $e \in F^n$. Hence $\Gamma(F)$ and F^n have the same idempotents.

Suppose $\Gamma(F)$ is a group and e is the identity of $\Gamma(F)$. Then e is also the identity of F^n. Let $x \in F^n$. There exists $x^{-1} \in \Gamma(F)$ such that $xx^{-1} = x^{-1}x = e$. Since $x^{-1} \in F^i$ for some $i = 1,..,n$ and $e \in F^n \cap F^nF^i$, by 3.2.3 we have $F^n = F^{n+i}$ and thus $F^i = F^n$, i.e. $x^{-1} \in F^n$. Hence F^n is a group. Conversely, suppose F^n is a group and e is the identity of F^n. For each $i = 1,..,n$ and $x \in F^i$, since

$F^iF^n = F^i$, there is $y \in F^i$ such that $ye = x$. Then $x = ye = ye^2 = xe$. Similarly $ex = x$ for all $x \in F^i$, $i = 1,\dots,n$. Thus e is the identity of $\Gamma(F)$. Let $x \in \Gamma(F)$. Then $x \in F^i$ for some i. If $i = n$, then x is invertible. Otherwise there are $y, z \in F^{n-i}$ such that $xy = e$ and $zx = e$. Again x is invertible. Hence $\Gamma(F)$ is a group.

3.2.6. Theorem.

μ is central if and only if F is normal in S. In this case $\Gamma(F)$ is a normal group in S.

Proof. If μ is central, then for any $x \in S$ we have $\delta_x\mu = \mu\delta_x$. Thus $xF = Fx$ for all $x \in S$, i.e. F is normal in S. Conversely, let F be normal in S. It suffices to show $\delta_x\mu = \mu\delta_x$ for all $x \in S$. We first assert that $\delta_x P(F) = P(F)\delta_x$ for all $x \in S$, where $P(F)$ is the set of probability measures on F. Let $\tilde{F} = \{\delta_a : a \in F\}$ and $\nu \in co(\tilde{F})$, the convex hull of $\tilde{F}$. Then $\nu = \sum t_i\delta_{a_i}$ which is a convex combination of elements in $\tilde{F}$. Since $xF = Fx$, for each a_i there is $b_i \in F$ such that $xa_i = b_ix$. Thus $\delta_x\nu = \delta_x\sum t_i\delta_{a_i} = \sum t_i\delta_{b_i}\delta_x \in co(\tilde{F})\delta_x$. Thus $\delta_x co(\tilde{F}) \subset co(\tilde{F})\delta_x$. Similarly $co(\tilde{F})\delta_x \subset \delta_x co(\tilde{F})$. Thus $\delta_x co(\tilde{F}) = co(\tilde{F})\delta_x$ and so $\delta_x \overline{co}(\tilde{F}) = \overline{co}(\tilde{F})\delta_x$ i.e. $\delta_x P(F) = P(F)\delta_x$. Since F is normal in S and μ^n is idempotent, then F^n is normal in S and simple. Thus F^n is a group and so $\Gamma(F)$ is a normal group in S. By [12, theorem 3.2] μ^n is central. We have $\mu\delta_x = \mu\mu^n\delta_x = \mu\delta_x\mu^n = \delta_x\nu\mu^n$ for some $\nu \in P(F)$. Since $\nu\mu^n = \nu\mu^n\mu^n = \mu^n\nu\mu^n = \mu^{n+1+n} = \mu$ by 3.2.2, thus $\mu\delta_x = \delta_x\mu$ for all $x \in S$, completing the proof.

3.2.7. Example.

$\Gamma(F)$ being normal in S is not sufficient to guarantee μ is central.

Consider S_3 and let $\mu = \delta_x \in P(S_3)$, where $x^3 = e$. Then $\Gamma(F) = \{e, x, x^2\}$ is a normal subgroup but μ is not central.

3.2.8. Corollary.

If $\Gamma(F)$ is a group and identical with the minimal ideal $K(S)$ of S, then μ is central.

Proof. By 3.2.6 it suffices to show that F is normal in S. Let e be the identity of $\Gamma(F)$ and $x \in S$. Then $xe \in \Gamma(F)$ and so $xe \in F^i$ for some $i = 1,..,n$. We have $\operatorname{supp} \mu^{2n-i}\delta_{xe} \subset F^{2n-i}F^i = F^n$. Since μ^n is an idempotent and F^n is a compact group by 3.2.5, then in view of 3.1.10 μ^n annihilates any $\nu \in P(S)$ supported in F^n. Thus $\mu^n(\mu^{2n-i}\delta_{xe}) = \mu^n$. Then $\mu^{i+1} = \mu^{i+1}\mu^n = \mu^{i+1}\mu^n(\mu^{2n-i}\delta_{xe}) = \mu\delta_{xe}$. Similarly $\mu^{i+1} = \delta_{xe}\mu$. Hence $\delta_{xe}\mu = \mu\delta_{xe}$ and so $xeF = Fxe$. Since e is the identity of $\Gamma(F)$ which is the minimal ideal of S, we have $xF = Fx$ for all $x \in S$, i.e. F is normal in S.

3.2.9. Theorem.

In P(S), $\Pi_n^1 = \Pi_1 \cup C\Pi_n^1$, $\Pi_n^2 = \Pi_1 \cup C\Pi_n^2$ and $\Pi_n^3 = \Pi_1 \cup C\Pi_n^3$.

Proof. It is clear that $\Pi_1 \cup C\Pi_n^1 \subset \Pi_n^1$ (1.2.2). Let $\mu \in \Pi_n^1$ and write $\omega = \frac{1}{n}(\mu + .. + \mu^n)$. Then $\omega^2 = \omega$ and $\omega^n\mu = \mu\omega^n = \omega$, i.e. $\omega \leq_1 \mu$. Since μ is primitive, we have $\omega = \mu$ or $\omega = \vartheta$, the zero of P(S). If $\omega = \mu$, then $\Gamma(F) = \operatorname{supp} \omega = \operatorname{supp} \vartheta = K(S)$ which is a group by 3.1.11. In view of 3.2.8 μ is central, i.e. $\mu \in C\Pi_n^1$.

Hence $\Pi_n^1 = \Pi_1 \cup C\Pi_n^1$. Since $\omega^n\mu = \mu^n\omega = \omega$ and $\omega\mu^n = \mu\omega^n = \omega$, by the same argument, we have $\Pi_n^2 = \Pi_1 \cup C\Pi_n^2$ and $\Pi_n^3 = \Pi_1 \cup C\Pi_n^3$.

3.2.10. <u>Corollary.</u>

In $P(S)$, $\Pi_n^1 = \Pi_n^2 = \Pi_n^3$.

<u>Proof.</u> Since $C\Pi_n^1 = C\Pi_n^2 = C\Pi_n^3$ by 1.2.23, the result follows by applying the previous proposition.

In the rest of this section, we use Π_n *to denote* Π_n^1 , Π_n^2 *or* Π_n^3.

3.2.11. <u>Theorem.</u>

$\mu \in \Pi_n$ if and only if $\mu^n \in \Pi_1$.

<u>Proof.</u> The "if" part is shown in 1.2.3. Now suppose $\mu \in \Pi_n$. By 3.2.9 $\mu \in \Pi_1$ or μ is central. If $\mu \in \Pi_1$, then $\mu^n = \mu \in \Pi_1$. If μ is central, then by 1.2.5 $\mu^n \in \Pi_1$.

3.2.12. <u>Theorem.</u>

$\mu \in C\Pi_n$ if and only if $\mu^n \in C\Pi_1$.

<u>Proof.</u> The "only if" part has been shown in 1.2.6. Now Suppose $\mu^n \in C\Pi_1$. Then $\mu^n \in \Pi_1$ and thus $\mu \in \Pi_n$. By 3.2.9 $\mu \in \Pi_1$ or μ is central. If $\mu \in \Pi_1$, then $\mu = \mu^n \in C\Pi_1$. In any case μ is central and so $\mu \in C\Pi_n$.

3.2.13. Proposition.

Let $\mu \in \Pi_n$ be normal. Then μ is central.

Proof. By 3.2.9 $\mu \in \Pi_1$ or μ is central. If $\mu \in \Pi_1$, then μ is a normal idempotent. By virtue of 1.1.5 μ is central.

3.2.14. Proposition.

If P(S) has no zero element, then :

(1) $\Pi_n = \Pi_1 = K(P)$, where K(P) is the minimal ideal of P(S).

(2) $C\Pi_n = \emptyset$.

Proof. (1) By Pym's result [27, §D theorem], $K(P) = \Pi_1 \subset \Pi_n$. Now take any $\mu \in \Pi_n$. Then $\mu^n \in \Pi_1 = K(P)$. Thus $\mu = \mu\mu^n \in K(P)$. Hence $\Pi_n \subset K(P)$. Consequently $\Pi_n = \Pi_1 = K(P)$.

(2) If $\mu \in C\Pi_n$, then $\mu^n \in C\Pi_1$ and so it suffices to show that $C\Pi_1 = \emptyset$. If $C\Pi_1 \neq \emptyset$, then let $\tau \in C\Pi_n \subset \Pi_1 = K(P)$. By [27, §D lemma] $\tau\nu\tau = \tau$ for all $\nu \in P(S)$, i.e. τ is a zero element of P(S), a contradiction. Hence $C\Pi_n = \emptyset$.

3.2.15. Theorem.

If P(S) has the zero ϑ, then the set $C\Pi_n$ is a closed semigroup. Moreover, it is a union of mutually disjoint groups, each of which is of the form $\{\nu,\dots,\nu^n\}$ (they may not be all distinct) for some $\nu \in C\Pi_n$, and the product of any two elements from different groups is ϑ.

Proof. By 1.2.12 and 2.1.11, we see that $C\Pi_n$ is a closed subsemigroup. In the proof of 1.2.12 we have $C\Pi_n = \bigcup_{\nu\in C\Pi_n} H(\nu^n)$. If

$H(\nu^n)$ is a singleton, then $H(\nu^n)$ is of the form $\{\nu,..,\nu^n\}$ in which $\nu = ... = \nu^n$. Suppose $H(\nu^n)$ contains more than one element, i.e. there exists $\tau \in H(\nu^n)$ with $\tau \neq \nu^n$. Then $\tau^n = \nu^n$ and $H(\nu^n) = H(\tau^n) \supset \{\tau,..,\tau^n\}$ by 1.2.7. Take any $\lambda \in H(\tau^n)$. If λ is idempotent, then $\lambda = \tau^n$. If λ is not idempotent, in the proof of 3.2.9 we see that both $\Gamma(\text{supp } \lambda)$ and $\Gamma(\text{supp } \tau)$ are equal to $K(S)$ which is a group. Thus supp λ intersects supp τ^i for some $i = 1,..,n$. Take $x \in \text{supp } \lambda \cap \text{supp } \tau^i$. As $\Gamma(\text{supp } \lambda)$ is a group, by 3.2.5 supp λ^n is a group. Since $\text{supp } \lambda^{n-1}\delta_x \subset \text{supp } \lambda^n$, by 3.1.10 $\lambda^n\lambda^{n-1}\delta_x = \lambda^n$. Thus $\lambda = \lambda\lambda^n = \lambda\lambda^n\lambda^{n-1}\delta_x = \lambda^n\delta_x$. Similarly $\tau^i = \tau^n\delta_x$. Since $\lambda^n = \tau^n$, we have $\lambda = \tau^i$. Consequently, $H(\tau^n) = \{\tau,..,\tau^n\}$ and the proof is complete.

We see in the following proposition, when S is a compact group, we can characterize non-zero primitive elements by means of their supports.

3.2.16. <u>Theorem.</u>

If S is a compact group, then $\mu \in \Pi_n \setminus \{\vartheta\}$ if and only if F^n is a maximal proper closed subgroup of S.

<u>Proof.</u> By 3.2.12, $\mu \in \Pi_n \setminus \{\vartheta\}$ if and only if $\mu^n \in \Pi_1 \setminus \{\vartheta\}$. In view of [17, theorem 1 : S is a compact group, $\mu \in \Pi_1 \setminus \{\vartheta\}$ if and only if supp μ is maximal proper closed subgroup of S], the result follows.

3.2.17. Examples.

(1) In general Π_n is not a semigroup.

Let $S = S_3$ be equipped with the discrete topology. Then S is a compact semigroup. Let $\mu = \frac{1}{2}(\delta_e + \delta_y)$ and $\nu = \frac{1}{2}(\delta_e + \delta_{xy})$, where $x^3 = y^2 = e$. Then supp $\mu = \{e, y\}$ and supp $\nu = \{e, xy\}$ are maximal proper closed subgroups of S. Hence the above proposition shows μ, $\nu \in \Pi_1 \setminus \{\vartheta\}$. However, supp $\mu\nu = \{e, y, xy, x\}$ is not a subgroup of S and so $\mu\nu \notin \Pi_1$.

(2) If P(S) has the zero, Π_n may contain Π_1 properly.

Let $S = Z_4$ be equipped with the discrete topology and $\mu = \frac{1}{2}(\delta_x + \delta_{x^3})$. Then $\mu^2 = \frac{1}{2}(\delta_e + \delta_{x^2})$ and $\mu^3 = \mu$. Thus $\mu \in I_2$. Since $= \{e, x^2\}$ is a maximal proper closed subgroup of S, $\mu^2 \in \Pi_1$ and so $\mu \in \Pi_2$. However, μ is not an idempotent and thus $\mu \notin \Pi_1$.

§3 Limit measures

Recall that for $\mu \in P(S)$, let $\mu_n = \frac{1}{n}(\mu + .. + \mu^n)$, the sequence $(\mu_n)_{n\in\mathbb{N}}$ converges to an idempotent measure $L(\mu)$ with $L(\mu)\mu = \mu L(\mu) = L(\mu)$ (2.2.4).

3.3.1. Proposition. [1, Theorem 3]

Let $\mu \in P(S)$. Then $\operatorname{supp} L(\mu) = K(\Gamma(\operatorname{supp} \mu))$.

3.3.2. Example.

In general, $L(\mu\nu) \neq L(\mu)L(\nu)$ and $L(\mu^n) \neq L(\mu)$.

Let $S = Z_2$ be equipped with the discrete topology and $\mu = \nu = \delta_x$. Then $L(\mu) = L(\nu) = \frac{1}{2}(\delta_x + \delta_e)$. But $L(\mu\nu) = L(\delta_e) = \delta_e$. Thus $L(\mu\nu) \neq L(\mu)L(\nu)$ and $L(\mu^2) \neq L(\mu)$.

The following proposition was given in [1] showing certain conditions to guarantee $L(\mu\nu) = L(\mu)L(\nu)$.

3.3.3. Proposition.

Suppose $\mu, \nu \in P(S)$ such that $L(\mu\nu)$ and $L(\nu)$ are central. Any one of the following conditions implies $L(\mu\nu) = L(\mu)L(\nu)$.

(1) The family $\{\operatorname{supp} \mu^n : n \in \mathbb{N}\}$ has the finite intersection property.

(2) Either $\operatorname{supp} \mu \subset \operatorname{supp} \mu^2$ or there exists an idempotent $e \in \operatorname{supp} \mu$.

We give other condition for $L(\mu\nu) = L(\mu)L(\nu)$.

3.3.4. Proposition.

Suppose $L(\mu)$ commutes with ν and $L(\mu\nu)$, and supp $L(\mu) \subset$ supp $L(\mu\nu)$. Then $L(\mu\nu) = L(\mu)L(\nu)$.

Proof. Observe that the sequence whose nth term is defined by $L(\mu)(\frac{1}{n}(\mu\nu + .. + (\mu\nu)^n))$, converges to $L(\mu)L(\mu\nu)$. On the other hand, $L(\mu)(\frac{1}{n}(\mu\nu + .. + (\mu\nu)^n)) = \frac{1}{n}(L(\mu)\mu\nu + ..+ L(\mu)(\mu\nu)^n) = \frac{1}{n}(L(\mu)\nu + .. + L(\mu)\nu^n) = L(\mu)(\frac{1}{n}(\nu + .. + \nu^n))$ where the second equality is given by the fact that $L(\mu)$ is idempotent and commutes with ν. Thus the sequence converges to $L(\mu)L(\nu)$ and so $L(\mu)L(\nu) = L(\mu)L(\mu\nu)$. Since supp $L(\mu) \subset$ supp $L(\mu\nu)$, by 3.1.7 we have $L(\mu\nu)L(\mu)L(\mu\nu) = L(\mu\nu)$. Hence $L(\mu\nu) = L(\mu\nu)L(\mu)L(\nu) = L(\mu)L(\mu\nu)L(\nu) = L(\mu)L(\nu)L(\nu) = L(\mu)L(\nu)$.

3.3.5. Corollary.

If supp $\mu \subset$ supp μ^2, then $L(\mu^n) = L(\mu)$ for all $n \in \mathbb{N}$.

Proof. The hypothesis gives supp $\mu \subset$ supp $\mu^2 \subset$ supp $\mu^3 \subset ...$ So we have $\Gamma(\text{supp } \mu) = \Gamma(\text{supp } \mu^2)$. Hence supp $L(\mu) = K(\Gamma(\text{supp } \mu)) = K(\Gamma(\text{supp } \mu^2)) =$ supp $L(\mu^2)$. We complete the proof by induction. First, it is clear that $L(\mu)$ commutes with μ and $L(\mu^2)$. By the above proposition, $L(\mu^2) = L(\mu)L(\mu) = L(\mu)$. Now we assume $L(\mu^{n-1}) = L(\mu)$. Again since $L(\mu)$ commutes with μ^{n-1} and $L(\mu^n)$, then $L(\mu^n) = L(\mu)L(\mu^{n-1}) = L(\mu)L(\mu) = L(\mu)$. Therefore $L(\mu^n) = L(\mu)$ for all $n \in \mathbb{N}$.

Recall that for an idempotent τ_α in P(S), we write $Q_\alpha = \{\mu \in P(S) : L(\mu) = \tau_\alpha\}$, and the family of such subsets gives a partition on P(S). In general Q_α may not be a subsemigroup while the closure $\overline{Q_\alpha} = \{\mu \in P(S) : \mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha\}$ must be a subsemigroup of P(S). In what follows, we investigate properties of supp Q_α and supp $\overline{Q_\alpha}$. We shall use the notation defined above.

3.3.6. Proposition.

supp Q_α = supp $\overline{Q_\alpha}$ is a closed subsemigroup of S.

Proof. By virtue of 3.1.6 we have supp Q_α = supp $\overline{Q_\alpha}$ = $\overline{\cup \{\text{supp } \mu : \mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha\}}$. By 2.1.4 it suffices to show that $\cup \{\text{supp}\mu : \mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha\}$, denoted by A, is a subsemigroup of S. Let x, y $\in$ A. Then x $\in$ supp μ and y $\in$ supp ν for some μ and ν in P(S) with $\mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha$ and $\nu\tau_\alpha = \tau_\alpha\nu = \tau_\alpha$. It follows that xy $\in$ (supp μ)(supp ν) = supp $\mu\nu$. Since $\mu\nu\tau_\alpha = \mu\tau_\alpha = \tau_\alpha$ and $\tau_\alpha\mu\nu = \tau_\alpha\nu = \tau_\alpha$, then xy $\in$ A. Thus A is a subsemigroup of S, completing the proof.

3.3.7. Proposition.

K(supp $\overline{Q_\alpha}$) = supp τ_α.

Proof. We first assert that supp τ_α is an ideal of supp $\overline{Q}_\alpha$ = $\overline{\cup \{\text{supp } \mu : \mu\tau_\alpha = \tau_\alpha = \tau_\alpha\}}$. If $\mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha$, then we have (supp μ)(supp τ_α) = (supp τ_α)(supp μ) = supp τ_α. Therefore ($\cup$ {supp μ : $\mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha$})(supp τ_α) = (supp τ_α)($\cup$ {supp μ : $\mu\tau_\alpha = \tau_\alpha\mu = \tau_\alpha$}) = supp τ_α. Taking closure for the sets in the above equalities, we get (supp $\overline{Q_\alpha}$)(supp τ_α) = (supp τ_α)(supp $\overline{Q_\alpha}$) =

supp τ_α. Thus supp τ_α is an ideal of supp $\overline{Q_\alpha}$. Since τ_α is idempotent, then supp τ_α is simple. Thus supp τ_α is the minimal ideal of supp $\overline{Q_\alpha}$, completing the proof.

3.3.8. <u>Example.</u>

Let $P(\text{supp } \overline{Q_\alpha}) = \{\mu \in P(S) : \text{supp } \mu \subset \text{supp } \overline{Q_\alpha}\}$. It is easy to see that $\overline{Q_\alpha} \subset P(\text{supp } \overline{Q_\alpha})$ and τ_α is the zero of $\overline{Q_\alpha}$. In general, $\overline{Q_\alpha}$ is properly contained in $P(\text{supp } \overline{Q_\alpha})$ and τ_α may not be the zero of $P(\text{supp } \overline{Q})$.

Let $S = \{a, b\}$ with the discrete topology and the multiplication table on S is defined by

	a	b
a	a	a
b	b	b

Then $P(S) = \{t\delta_a + (1-t)\delta_b : 0 \leq t \leq 1\}$. It is clear that every element in $P(S)$ is idempotent and $P(S)$ has no zero. Let $\tau = \frac{1}{2}(\delta_a + \delta_b)$. Then $Q = \{\tau\}$ and $\overline{Q} = Q$ which is properly contained in $P(\text{supp } \tau) = P(S)$.

3.3.9. <u>Theorem.</u>

If supp τ_α is a group, then τ_α is the zero of $P(\text{supp } \overline{Q_\alpha})$ and $\overline{Q_\alpha} = P(\text{supp } \overline{Q_\alpha})$.

<u>Proof.</u> Since τ_α is idempotent and supp τ_α is a compact group, by 3.1.10 τ_α is the Haar measure of supp τ_α. In view of 3.3.7 we have $K(\text{supp } \overline{Q_\alpha}) = \text{supp } \tau_\alpha$ and by 3.1.11 τ_α is the zero of $P(\text{supp } \overline{Q_\alpha})$, then $\overline{Q_\alpha} = P(\text{supp } \overline{Q_\alpha})$.

3.3.10. Corollary.

If S is commutative, then τ_α is the zero of $P(\text{supp } \overline{Q_\alpha})$ and $\overline{Q_\alpha} = P(\text{supp } \overline{Q_\alpha})$.

Proof. Since S is commutative and supp τ_α is simple, then supp τ_α is a group. By the above theorem the result follows.

3.3.11. Theorem.

τ_α is the zero of P(S) if and only if Q_α is an ultra-convex set in P(S).

Proof. Let τ_α be the zero of P(S). We want to show $L(t\mu + (1-t)\nu) = \tau_\alpha$ for any $\mu \in Q_\alpha$, $\nu \in P(S)$ and $0 < t < 1$. Suppose $L(t\mu + (1-t)\nu) = \tau_\beta$ for some idempotent τ_β. Then for any $n \in \mathbb{N}$, $\tau_\beta = \tau_\beta(t\mu + (1-t)\nu)^n\tau_\beta = t^n\tau_\beta\mu^n\tau_\beta + (1-t^n)\tau_\beta\omega\tau_\beta$ for some $\omega \in P(S)$. By 2.2.2 we have $\tau_\beta = \tau_\beta\mu^n\tau_\beta$ and thus $\tau_\beta = \tau_\beta\eta\tau_\beta$ for $\eta \in \overline{co}(\Gamma(\mu))$. Therefore $\tau_\beta = \tau_\beta\tau_\alpha\tau_\beta$. That τ_α is the zero of P(S) gives $\tau_\beta = \tau_\alpha$. Hence $L(t\mu + (1-t)\nu) = \tau_\alpha$ and so Q_α is ultra-convex. Conversely, suppose Q_α is ultra-convex. By 2.2.15 Q_α is dense and by 2.2.12 τ_α is the zero of P(S).

3.3.12. Theorem.

If τ_α is the zero of P(S), then supp Q_α = S. Conversely, if supp Q_α = S and supp τ_α is a group, then τ_α is the zero of P(S).

Proof. Suppose τ_α is the zero of P(S); then Q_α is ultra-convex. Fix $\mu \in Q_\alpha$. For any $x \in S$, let $\nu = \frac{1}{2}(\mu + \delta_x)$ which is belonged to Q_α. Then $x \in \text{supp } \nu \subset \text{supp } Q_\alpha$. Thus supp Q_α = S. Conversely, by the hypothesis that supp τ_α is a group, τ_α is the zero of $P(\text{supp } \overline{Q_\alpha}) = P(\text{supp } Q_\alpha) = P(S)$ by virtue of 3.3.9. The proof is complete.

3.3.13. Examples.

To fulfil the remark in section 2 of Chapter 2, we give examples showing that :

(1) If S has infinite number of idempotents, then τ_α being a minimal idempotent may not imply Q is open.

Consider P(S) described in 3.3.8. Since every element in P(S) is idempotent and $\mu\nu = \nu$ for $\mu, \nu \in P(S)$, no two elements are related by the order $\leq$. Hence each element in P(S) is both maximal and minimal. Let τ_α be any element in P(S). Then $Q_\alpha = \{\tau_\alpha\}$ is closed and Q_α is not open, otherwise the topology of P(S) is discrete which contradicts to P(S) being compact.

(2) Q_α need not be a semigroup.

Let $S = Z_2$ be equipped with the discrete topology. Then we have $\delta_x \in Q_1 = P(S) \setminus \{\delta_e\}$ and $\delta_x^2 = \delta_e \in Q_2 = \{\delta_e\}$. Thus Q_1 is not a semigroup.

§4 Nilpotent measures

In section 2 of Chapter 2, we have seen that the set of all nilpotent elements of a compact affine semigroup is ultra-convex and so it is convex and dense in the semigroup. In this section we assume that P(S) has zero element. i.e. K(S) is a compact group, and we can talk about nilpotent measures in P(S). Our objective of this section is to characterize nilpotent measures in terms of their supports. We first consider a special case that K(S) is a singleton {0}, and so 0 is the zero of S while the Dirac measure $\vartheta = \delta_0$ is the zero of P(S).

3.4.1. Definitions.

In a compact semigroup S with zero 0, we have defined that an element x is nilpotent if $x^n \to 0$ as $n \to \infty$. Here we extend this definition to subsets of the semigroup. $A \subset S$ is called nilpotent if $A^n \to 0$ as $n \to \infty$, i.e. for any neighbourhood U of 0, there exists a positive integer k such that $A^n \subset U$ for $n \geq k$. The set $A \subset S$ is said to be quasi-nilpotent if $0 \in \Gamma(A)$.

It is clear that if $A \subset S$ is nilpotent, then A is quasi-nilpotent. In general the converse is not true. For example, let S = [0,1] with the usual multiplication and the usual topology. Take A = [0,1). Then it is obvious that A is quasi-nilpotent but not nilpotent. However, we have the positive result for singleton sets.

3.4.2. Theorem.

Suppose S is a compact semigroup with zero 0. Then $x \in S$ is nilpotent if and only if quasi-nilpotent.

Proof. It suffices to show that x is nilpotent if it is quasi-nilpotent. Since $0 \in \Gamma(x)$, then $K(\Gamma(x)) = \{0\}$. Thus the sequence (x^n) has a unique cluster point by 2.1.7. Therefore we have $x^n \to 0$ as $n \to \infty$, completing the proof.

3.4.3. Lemma.

Let S be a compact semigroup and $A \subset P(S)$. Then $\operatorname{supp} \Gamma(A) = \operatorname{supp} \bigcup_{n=1}^{\infty} A^n = \Gamma(\operatorname{supp} A)$.

Proof. The first equality has been shown in 3.1.6. We assert the second equality. Since $\Gamma(A) \supset A^n$ for all $n \in \mathbb{N}$, then $\operatorname{supp} \bigcup_{n=1}^{\infty} A^n \supset \operatorname{supp} A^n = (\operatorname{supp} A)^n$ and so $\operatorname{supp} \bigcup_{n=1}^{\infty} A^n \supset \Gamma(\operatorname{supp} A)$. On the other hand, take any $\mu \in \bigcup_{n=1}^{\infty} A^n$, then $\mu \in A^n$ for some n. Therefore we have $\operatorname{supp} \mu \subset \operatorname{supp} A^n = (\operatorname{supp} A)^n \subset \Gamma(\operatorname{supp} A)$. It follows that $\operatorname{supp} \bigcup_{n=1}^{\infty} A^n \subset \Gamma(\operatorname{supp} A)$ and consequently the result follows.

3.4.4. Theorem.

Suppose S is a compact semigroup with zero 0. A subset $A \subset P(S)$ is quasi-nilpotent if and only if $\operatorname{supp} A$ is quasi-nilpotent in S.

Proof. Suppose A is quasi-nilpotent, i.e. the zero $\vartheta = \delta_0$ of $P(S)$ is contained in $\Gamma(A)$. By the above lemma, $0 \in \Gamma(\operatorname{supp} A)$. Thus supp A is quasi-nilpotent. Conversely, suppose that supp A is quasi-nilpotent in S, i.e. $0 \in \Gamma(\text{supp } A)$. Thus $\{0\} = K(\Gamma(\text{supp } A)) = K(\text{supp } \Gamma(A)) = \text{supp } K(\Gamma(A))$, where the last equality is guaranteed by 3.1.6. Hence $K(\Gamma(A)) = \{\vartheta\}$ and so $\vartheta \in \Gamma(A)$, i.e. A is quasi-nilpotent.

Combining 3.4.2 and 3.4.4, we have the following theorem.

3.4.5. <u>Theorem.</u>

Let S be a compact semigroup with zero element. A measure $\mu \in P(S)$ is nilpotent if and only if supp μ is quasi-nilpotent in S.

Now we consider the general case. *In what follows,* S *always denotes a compact semigroup whose minimal ideal* K(S) *is a compact group, unless the contrary is explicitly specified.* Then the (normalized) Haar measure ϑ of K(S) is the zero of P(S).

3.4.6. <u>Proposition.</u>

A measure $\mu \in P(S)$ with supp $\mu \supset K(S)$ is nilpotent.

<u>Proof.</u> By 2.1.7 $K(\Gamma(\mu))$ is a group. Let τ be the identity of $K(\Gamma(\mu))$. Then $\text{supp } \tau \subset \text{supp } K(\Gamma(\mu)) = K(\text{supp } \Gamma(\mu)) = K(\Gamma(\text{supp } \mu)) = K(S)$ where the first equality is given by 3.1.6, the second by 3.4.3 and the third is due to supp $\mu \supset K(S)$ and by 1.1.3. Thus $\text{supp } \mu\tau = (\text{supp } \mu)(\text{supp } \tau) \supset (K(S))(\text{supp } \tau) = K(S)$. That $\mu\tau \in K(\Gamma(\mu))$ gives $\text{supp } \mu\tau \subset \text{supp } K(\Gamma(\mu)) = K(S)$. Hence $\text{supp } \mu\tau = K(S)$. As $K(\Gamma(\mu))$ is a group, there exists $\nu \in K(\Gamma(\mu))$

such that $(\mu\tau)\nu = \tau$. Thus supp τ = (supp $\mu\tau$)(suppν) = K(S)(supp ν) = K(S) since supp $\nu \subset$ supp $K(\Gamma(\mu))$ = K(S). By virtue of 3.1.10, τ is the Haar measure of K(S), i.e. $\tau = \vartheta$. Hence $K(\Gamma(\mu)) = \{\vartheta\}$. Therefore the sequence (μ^n) has only one cluster point ϑ and so $\mu^n \to \vartheta$, i.e. μ is nilpotent.

3.4.7. <u>Corollary.</u>

Let $\mu \in P(S)$ and supp $\mu^n \supset K(S)$ for some n. Then μ is nilpotent.

<u>Proof.</u> Let U be an arbitrary neighbourhood of ϑ. Since $\mu^i\vartheta = \vartheta$, there is a neighbourhood V of ϑ such that $\mu^i V \subset U$ for i = 1,..,n. By the above proposition μ^n is nilpotent, there exists a positive integer $k_\circ$ such that $(\mu^n)^k \in V$ for all $k \geq k_\circ$. Thus $\mu^i\mu^{nk} = \mu^{nk+i} \in U$ where i = 1,..,n and $k \geq k_\circ$. Hence for $j > nk_\circ$, $\mu^j \in U$. Thus μ is nilpotent.

3.4.8. <u>Example.</u>

The converse of 3.4.6 is not true.

Let S = [0,1] with the usual multiplication and the usual topology. Take $\mu = \delta_{\frac{1}{2}}$ which is clearly nilpotent but supp $\delta_{\frac{1}{2}} = \{\frac{1}{2}\}$ does not contain K(S) = {0}.

3.4.9. <u>Definition.</u>

Given $\mu \in P(S)$, we have seen in the previous section (3.3.1) that the sequence $(\frac{1}{n}(\mu + \ldots + \mu^n))$ must converge to an idempotent measure $L(\mu)$ with supp $L(\mu) = K(\Gamma(\text{supp } \mu))$. The measure μ is termed mean-nilpotent if $L(\mu) = \vartheta$.

3.4.10. Proposition.

A nilpotent measure is mean-nilpotent.

Proof. Let $\mu \in P(S)$ be nilpotent. By 2.2.5 $L(\mu) = \vartheta$, i.e. μ is mean-nilpotent.

3.4.11. Example.

A mean-nilpotent may not be nilpotent.

Let $S = Z_2$ be equipped with the discrete topology. Take $\mu = \delta_x$. Then $\text{supp } L(\mu) = K(\Gamma(\text{supp } \mu)) = K(\Gamma(x)) = K(S) = S$. Since $L(\mu)$ is idempotent and S is a group, by 3.1.10 $L(\mu) = \vartheta$ which is the zero of P(S), i.e. μ is mean-nilpotent. But μ is not nilpotent since the sequence (μ^n) does not converge.

3.4.12. Proposition.

Suppose S is a compact semigroup with zero 0. A measure $\mu \in P(S)$ is nilpotent if and only if mean-nilpotent.

Proof. Since $\text{supp } L(\mu) = K(\Gamma(\text{supp } \mu))$, then it is easy to see that $L(\mu) = \delta_0$ if and only if $0 \in \Gamma(\text{supp } \mu)$, i.e. μ is quasi-nilpotent. In view of 3.4.5, the result follows.

3.4.13. Theorem.

A measure $\mu \in P(S)$ is mean-nilpotent if and only if $\Gamma(\text{supp } \mu) \supset K(S)$.

Proof. Suppose $L(\mu) = \vartheta$. $\Gamma(\text{supp } \mu) \supset K(\Gamma(\text{supp } \mu)) = \text{supp } L(\mu) = \text{supp } \vartheta = K(S)$. Conversely, suppose $\Gamma(\text{supp } \mu) \supset K(S)$. By 1.1.3 $K(\Gamma(\text{supp } \mu)) = K(S)$, thus $\text{supp } L(\mu) = K(S)$. Hence $L(\mu) = \vartheta$, i.e. μ is mean-nilpotent.

3.4.14. Corollary.

If $\mu^n \in P(S)$ is mean-nilpotent for some n, then μ is mean-nilpotent.

Proof. Since $\Gamma(\text{supp } \mu) \supset \Gamma(\text{supp } \mu^n)$, the result follows by applying the previous proposition.

3.4.15. Notation.

Let $\mu \in P(S)$. The set $\{x \in S$: each neighbourhood of x meets supp μ^n from some n on$\}$ is denoted by $\Lambda(\text{supp } \mu)$ and is called lim inf supp μ^n in [19].

As a consequence of [19, corollary 1.1], we have the following lemma.

3.4.16. Lemma.

Let $\text{supp } \mu \subset K(S)$. Then μ is nilpotent if and only if $\Lambda(\text{supp } \mu) = K(S)$.

3.4.17. Theorem.

A measure $\mu \in P(S)$ is nilpotent if and only if $\Lambda(\text{supp } \mu) \supset K(S)$.

Proof. Suppose μ is nilpotent. Let $x \in K(S)$. If $x \notin \Lambda(\text{supp } \mu)$, then there exists a neighbourhood U of x with $U \cap \text{supp } \mu^n = \emptyset$ for infinitely many n. Since $x \in K(S) = \text{supp } \vartheta$, there exists $f \in C(S)$ with $f(S \setminus U) = 0$ and $\vartheta(f) \neq 0$. Since $U \cap \text{supp } \mu^n = \emptyset$, we have $f(\text{supp } \mu^n) = 0$ for infinitely many n. Thus $\mu^n(f) = 0$ for infinitely many n. As $\mu^n \to \vartheta$, i.e. $\mu^n(f) \to \vartheta(f)$ for all $f \in C(S)$. Then $\vartheta(f) = 0$, a contradiction. Thus $\Lambda(\text{supp } \mu) \supset K(S)$.

Conversely, suppose $\Lambda(\text{supp } \mu) \supset K(S)$. Let e be the identity of the group K(S) and consider the sequence $((\mu\delta_e)^n)$. We assert that $\Lambda(\text{supp } \mu\delta_e) = K(S)$. Let $x \in \Lambda(\text{supp } \mu\delta_e)$ and U be a neighbourhood of x. Then there exists a positive integer $n_\circ$ such that $U \cap \text{supp}(\mu\delta_e)^n = (\text{supp } \mu\delta_e)^n = ((\text{supp } \mu)e)^n \subset K(S)$ for $n \geq n_\circ$. Thus each neighbourhood of x intersects K(S). As K(S) is closed, we have $x \in K(S)$, i.e. $\Lambda(\text{supp } \mu\delta_e) \subset K(S)$. Now let $x \in K(S)$ and V be a neighbourhood of x. Since $xe = x$, there exists a neighbourhood W of x such that $We \subset V$. Since $x \in K(S) \subset \Lambda(\text{supp } \mu)$, there exists a positive integer $n_\circ$ such that $W \cap \text{supp } \mu^n \neq \emptyset$ for all $n \geq n_\circ$. Thus $\emptyset \neq (W \cap \text{supp } \mu^n)e \subset We \cap (\text{supp } \mu^n)e = We \cap \text{supp } (\mu\delta_e)^n$ for all $n \geq n_\circ$, we have $x \in \Lambda(\text{supp } \mu\delta_e)$, i.e. $K(S) \subset \Lambda(\text{supp } \mu\delta_e)$. Consequently, $\Lambda(\text{supp } \mu\delta_e) = K(S)$.

Since $\text{supp } \mu\delta_e = (\text{supp } \mu)e \subset K(S)$, by the above lemma $\mu\delta_e$ is nilpotent. Let τ be the unique idempotent of $\Gamma(\mu)$. Then there is a subnet (μ^α) of the sequence (μ^n) converging to τ. Since e is the identity of K(S), δ_e is the identity of measures supported in K(S). Hence $(\mu\delta_e)^\alpha = \mu^\alpha\delta_e \to \tau\delta_e$ and thus $\tau\delta_e = \vartheta$. Since $\text{supp } \tau \subset \text{supp } K(\Gamma(\mu)) \subset K(\text{supp } \Gamma(\mu)) \subset K(S)$, it follows that $\tau\delta_e = \tau$. Thus $\tau = \vartheta$. Hence $K(\Gamma(\mu)) = \{\vartheta\}$. Since ϑ is the unique cluster point of $\Gamma(\mu)$, then μ is nilpotent, completing the proof.

3.4.18. Example.

The inclusion in the above theorem may be proper.

Let $S = [0,1]$ with the usual multiplication and the usual topology. Let $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Then μ is nilpotent and $\Lambda(\text{supp } \mu) = \{0, 1\}$ contains $K(S) = \{0\}$ properly.

3.4.19. Corollary.

Let μ, $\nu \in P(S)$ with supp $\mu \subset$ supp ν. If μ is nilpotent, then ν is nilpotent.

Proof. Since supp $\mu \subset$ supp ν, we get supp $\mu^n \subset$ supp ν^n for all n. Thus $\Lambda(\text{supp } \mu) \subset \Lambda(\text{supp } \nu)$. Since μ is nilpotent, then $K(S) \subset \Lambda(\text{supp } \mu)$ and so $K(S) \subset \Lambda(\text{supp } \nu)$. Thus ν is nilpotent.

3.4.20. Remark.

Proposition 3.4.6 is a simple consequence of the above theorem.

§5 Bonds in P(S)

We have seen in the previous sections that given a subset Λ in P(S), how its support supp Λ being a subset in S relates to it. In this section we study in the opposite direction, in the sense that given a subset A in S, we consider a subset $\beta(A)$ in P(S), which is named a bond over A, corresponding to A by means of supports.

3.5.1. Definition.

A subset Λ of P(S) is called a bond in P(S) if there exists a non-empty subset A of S such that $\Lambda = \{\mu \in P(S) : \text{supp}\ \mu \cap A \neq \emptyset\}$; we call Λ the bond over the underlying set A and write $\Lambda = \beta(A)$.

3.5.2. Remarks.

(1) Given a bond Λ in P(S), its underlying set is unique.

Proof. Suppose $\Lambda = \beta(A_1) = \beta(A_2)$ for $A_1 \neq A_2$ in S. Without loss of generality, we assume that there exists $x \in A_1 \setminus A_2$. Therefore $\delta_x \in \beta(A_1)$ and $\delta_x \notin \beta(A_2)$ contradicts to $\beta(A_1) = \beta(A_2)$. Hence $A_1 = A_2$, i.e. the underlying set is unique.

(2) A subset Λ of P(S) may not be a bond.

For example, let S = [0,1] with the usual multiplication and the usual topology. Let $\Lambda = \{\delta_0\}$. If Λ is a bond , then its underlying set A must contain 0. It is clear that $\beta(A)$ contains Λ properly. Hence Λ is not a bond in P(S).

(3) Given a subset A of S, there is a unique bond in P(S) corresponding to A, namely $\beta(A)$.

3.5.3. Proposition.

Let $A, A_1, A_2, A_i \subset S$ for $i \in I$. Then :

(1) $A_1 \subset A_2$ if and only if $\beta(A_1) \subset \beta(A_2)$.

(2) $\beta(A_1)\beta(A_2) \subset \beta(A_1A_2)$.

(3) A is an ideal of S if and only if $\beta(A)$ is an ideal of P(S).

(4) $\beta(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \beta(A_i)$; hence, a union of bonds is a bond.

(5) $\beta(\bigcap_{i\in I} A_i) \subset \bigcap_{i\in I} \beta(A_i)$, if $\bigcap_{i\in I} A_i \neq \emptyset$.

Proof. (1) We need only show the "if" part. Let $x \in A_1$. Then $\delta_x \in \beta(A_1) \subset \beta(A_2)$. Thus $x \in A_2$, i.e. $A_1 \subset A_2$.

(2) Let $\mu\nu \in \beta(A_1)\beta(A_2)$ with $\mu \in \beta(A_1)$ and $\nu \in \beta(A_2)$. Then supp $\mu \cap A_1 \neq \emptyset$ and supp $\nu \cap A_2 \neq \emptyset$. Thus supp $\mu\nu \cap A_1A_2 \neq \emptyset$ and so $\mu\nu \in \beta(A_1A_2)$. Therefore $\beta(A_1)\beta(A_2) \subset \beta(A_1A_2)$.

(3) Suppose A is an ideal of S. Then $\beta(A)P(S) = \beta(A)\beta(S) \subset \beta(AS) \subset \beta(A)$. Similarly $P(S)\beta(A) \subset \beta(A)$. Thus $\beta(A)$ is an ideal of P(S). Conversely, suppose $\beta(A)$ is an ideal of P(S). Let $x \in A$ and $y \in S$. Then $\delta_{xy} = \delta_x\delta_y \in \beta(A)$. Thus $xy \in A$. Similarly $yx \in A$. Hence A is an ideal of S.

(4) Since $A_i \subset \bigcup_{i\in I} A_i$ for all $i \in I$. By (1) above $\beta(A_i) \subset \beta(\bigcup_{i\in I} A_i)$. Thus $\bigcup_{i\in I}\beta(A_i) \subset \beta(\bigcup_{i\in I} A_i)$. For the other inclusion, take $\mu \in \beta(\bigcup_{i\in I} A_i)$. Then supp $\mu \cap A_i \neq \emptyset$ for some $i \in I$. Thus $\mu \in \beta(A_i) \subset \bigcup_{i\in I}\beta(A_i)$ and so $\beta(\bigcup_{i\in I} A_i) \subset \bigcup_{i\in I} \beta(A_i)$. Consequently, the result follows.

(5) Since $\bigcap_{i\in I} A_i \subset A_i$ for all $i \in I$. Then $\beta(\bigcap_{i\in I} A_i) \subset \beta(A_i)$ for all $i \in I$. The result is immediate.

3.5.4. Remarks.

(1) Inclusion in 3.5.3(2) may be proper.

For example, let $S = \{a, b\}$ with the discrete topology and the multiplication defined by

	a	b
a	a	a
b	a	a

Let $A_1 = A_2 = \{b\}$. Then $\beta(A_1) = \beta(\{b\}) = \{t\delta_a + (1-t)\delta_b : 0 \le t < 1\}$ and $\beta(A_1A_2) = \beta(\{a\}) = \{ t\delta_a + (1-t)\delta_b : 0 < t \le 1\}$, but $\beta(A_1)\beta(A_2) = \{\delta_a\}$. Thus $\beta(A_1A_2)$ contains $\beta(A_1)\beta(A_2)$ properly.

(2) Inclusion in 3.5.3(5) may be proper too.

For example, let $S = \{a, b, c\}$ with the discrete topology and any multiplication such that S is a semigroup. Let $A_1 = \{a, b\}$ and $A_2 = \{b, c\}$. Then $\beta(A_1) = P(S) \setminus \delta_c$, $\beta(A_2) = P(S) \setminus \delta_a$ and $\beta(A_1 \cap A_2) = \beta(\{b\}) = P(S) \setminus \{t\delta_a + (1-t)\delta_c : 0 \le t \le 1\}$, but $\beta(A_1) \cap \beta(A_2) = P(S) \setminus \{\delta_b , \delta_c\}$. Thus $\beta(A_1) \cap \beta(A_2)$ contains $\beta(A_1 \cap A_2)$ properly.

(3) The intersection of a finite number of bonds is always non-empty.

Proof. Let $a_i \in A_i$ for $i = 1,..,n$ and $\mu = \frac{1}{n}(\delta_{a_1} + .. + \delta_{a_n})$. Then $\mu \in \beta(A_i)$ for $i = 1,..,n$. Thus $\mu \in \bigcap_{i\in I}^{n} \beta(A_i)$.

(4) The intersection of infinitely many bonds may be empty.

Note that if $\mu \in \bigcap_{a\in S}\beta(\{a\})$. Then supp $\mu = S$. Thus $\bigcap_{a\in S} \beta(\{a\}) = \emptyset$ if P(S) has no measure with supports equal to S. For example, we refer to [24, Example].

3.5.5. Theorem.

Let $A \subset S$. Then :

(1) $\beta(A)$ is ultra-convex.

(2) $S = \bigcup_{\mu\in\beta(A)} \operatorname{supp} \mu$.

Proof. (1) Let $\Lambda = \{t\mu + (1-t)\nu : \mu \in \beta(A), \nu \in P(S), 0 < t < 1\}$. It suffices to show that $\beta(A) = \Lambda$. It is clear that $\beta(A) \subset \Lambda$. Now let $\omega = t\mu + (1-t)\nu \in \Lambda$. Then $\operatorname{supp} \omega = (\operatorname{supp} \mu) \cup (\operatorname{supp} \nu)$ intersects A and so $\omega \in \beta(A)$. Thus $\beta(A) = \Lambda$.

(2) It needs only to show that $S \subset \bigcup_{\mu\in\beta(A)} \operatorname{supp} \mu$. Let $x \in S$ and take $a \in A$. Then $\mu = \frac{1}{2}(\delta_x + \delta_a) \in \beta(A)$. Thus $x \in \bigcup_{\mu\in\beta(A)} \operatorname{supp} \mu$ and the result follows.

3.5.6. Remark.

By the above proposition, a bond in P(S) is always ultra-convex, but an ultra-convex set in P(S) may not be a bond.

Suppose S is a compact semigroup with the zero 0. Let N(P(S)) and N(S) be the sets of all nilpotents in P(S) and S respectively. By 2.1.17, N(P(S)) is ultra-convex. If N(P(S)) is a bond, then $N(P(S) = \beta(A)$ for some $A \subset S$. We claim that $A = N(S)$. Let $x \in A$. Then $\delta_x \in \beta(A) = N(P(S))$. Thus $(\delta_x)^n \to \delta_0$, i.e. $\delta_{x^n} \to \delta_0$. Hence $x^n \to 0$, i.e. $x \in N(S)$, and so $A \subset N(S)$. Now let $\mu \in \beta(N(S))$, i.e. $\operatorname{supp} \mu \cap N(S) \neq \emptyset$. Then it is easy to see that μ is quasi-nilpotent and thus nilpotent by 3.4.2, i.e. $\mu \in N(P(S))$. Hence $\beta(N(S)) \subset N(P(S)) = \beta(A)$ and so $N(S) \subset A$. Consequently $A = N(S)$, i.e. $N(P(S)) = \beta(N(S))$. Therefore N(P(S)) cannot be a bond if $N(P(S)) \neq \beta(N(S))$.

Let $S = \{a, b, c, d\}$ with the discrete topology and the multiplication defined by

	a	b	c	d
a	a	a	a	a
b	a	a	b	a
c	a	a	c	a
d	a	b	b	d

Let $\mu = \frac{1}{2}(\delta_c + \delta_d)$. Then $\mu \in N(P(S))$. Since $N(S) = \{a, b\}$, it follows that supp $\mu \cap N(S) = \emptyset$, i.e. $\mu \notin \beta(N(S))$ and so $N(P(S))$ is not a bond.

3.5.7. Theorem.

Let $A \subset S$. A is a completely prime ideal of S if and only if $\beta(A)$ is a completely prime ideal of $P(S)$.

Proof. Suppose A is a completely prime ideal of S. Then $\beta(A)$ is an ideal of $P(S)$. Let $\mu, \nu \in P(S)$ with $\mu\nu \in \beta(A)$. Then $(\text{supp } \mu)(\text{supp}\nu) \cap A \neq \emptyset$; there exist $a \in$ supp μ and $b \in$ supp ν such that $ab \in A$. Since A is completely prime, we have $a \in A$ or $b \in A$, i.e. supp $\mu \cap A \neq \emptyset$ or supp $\nu \cap A \neq \emptyset$. Thus $\mu \in \beta(A)$ or $\nu \in \beta(A)$, and so $\beta(A)$ is a completely prime ideal of $P(S)$. Conversely, suppose $\beta(A)$ is a completely prime ideal of S and so A is an ideal of S. Let $x, y \in S$ with $xy \in A$. Then we have $\delta_x\delta_y = \delta_{xy} \in \beta(A)$. Thus $\delta_x \in \beta(A)$ or $\delta_y \in \beta(A)$, i.e. $x \in A$ or $y \in A$. Hence A is a completely prime ideal of S.

3.5.8. Theorem.

Let $A \subset S$. If $\beta(A)$ is a prime ideal of $P(S)$, then A is a prime ideal of S.

Proof. Since $\beta(A)$ is an ideal of $P(S)$, then A is an ideal of S. Let I and J be ideals of S with $IJ \subset A$. Then $\beta(IJ) \subset \beta(A)$ and so $\beta(I)\beta(J) \subset \beta(A)$. Since I and J are ideals of S, then $\beta(I)$ and $\beta(J)$ are ideals of $P(S)$. As $\beta(A)$ is prime, we have either $\beta(I) \subset \beta(A)$ or $\beta(J) \subset \beta(A)$ and so $I \subset A$ or $J \subset A$. Hence A is a prime ideal of S.

3.5.9. Theorem.

Let $A \subset S$. A is open if and only if $\beta(A)$ is open.

Proof. Suppose A is open. Then $S \setminus A$ is compact. By net argument, we see that $P(S \setminus A) = \{\mu \in P(S) : \text{supp } \mu \subset S \setminus A\}$ is compact. Thus $\beta(A) = P(S) \setminus P(S \setminus A)$ is open. Conversely, suppose $\beta(A)$ is open. It is equivalent to show that $S \setminus A$ is closed. Let $x \in \overline{S \setminus A}$. Then there exists a net (x_α) in $S \setminus A$ such that $x_\alpha \to x$. Thus $\delta_{x_\alpha} \in P(S \setminus A)$ with $\delta_{x_\alpha} \to \delta_x$. Since $P(S \setminus A) = P(S) \setminus \beta(A)$ is closed, then $\delta_x \in P(S \setminus A)$. Thus $x \in S \setminus A$, i.e. $S \setminus A$ is closed, completing the proof.

BIBLIOGRAPHY

[1] H.L.Chow, Limit measures on compact semitopological semigroups, Math. Scand. 33(1973), 375-381.

[2] ________, On supports of semigroups of measures, Proc. Edinburgh Math. Soc. (2)19(1974), 31-33.

[3] ________, Probability measures of finite order, J. London Math. Soc. (2)8(1974), 176-178.

[4] ________, Elements of finite order in semigroups, Math. Japon. (3)20(1975), 219-229.

[5] ________, Quasi-nilpotent sets in semigroups, Proc. Amer. Math. Soc. 52(1975), 393-397.

[6] ________, Nilpotent measures on compact semigroups, Bull. Austral. Math. Soc. 12(1975), 149-153.

[7] ________, On compact affine semigroups, Semigroup Forum, 11(1975), 146-152.

[8] ________, Limits of averaged product sequences in compact affine semigroups, Glasnik Matematicki (12)32 (1977), 261.

[9] ________, Radicals and semiprime ideals in semigroups, Acta Math. Hungar. 29(1977), 49-50.

[10] ________, A note on nilpotent measures, Arch. Math. 31 (1978), 286-287.

[11] ________, A note on bonds in semigroups of measures, Nanta Math. (1)9(1976), 89-91.

[12] S.T.L.Choy, Idempotent measures on compact semigroups, Proc. London Math. Soc. (3)20(1970), 717-733.

[13] S.T.L.Choy, On a limit theorem of measures, Math. Scand. 29(1971), 256-258.

[14] S.T.L.Choy, B.Dummigan and J.Duncan, On compact normal semigroups, Proc. Edinburgh Math. Soc. (2) 16(1969), 333-338.

[15] A.H.Clifford and G.B.Preston, The algebraic theory of semigroups, Vol.1 (Amer. Math. Soc. 1961).

[16] H.Cohen and H.S.Collins, Affine semigroups, Trans. Amer. Math. Soc. 93(1959), 97-113.

[17] H.S.Collins, Primitive idempotents in the semigroup of measures, Duke Math. J. 27(1960), 397-400.

[18] ________, The kernel of a semigroup of measures, Duke Math. J. 28(1961), 387-392.

[19] ________, Convergence of convolution iterates of measures, Duke Math. J. 29(1962), 259-264.

[20] J.Duncan, Primitive idempotent measures on compact semigroups, Proc. Edinburgh Math. Soc. (2) 17(1970), 95-103.

[21] N.Dunford and J.L.Schwartz, Linear operators, Part 1, (Interscience, New York, 1958).

[22] I.Glicksberg, Convolution semigroups of measures, Pacific J. Math. 9(1959), 51-67.

[23] R.J.Koch, On monothetic semigroups, Proc. Amer. Math. Soc. 8(1957), 397-401.

[24] Y.F.Lin, Not necessarily abelian convolution semigroups of probability measures, Math. Zeitschr. 91(1966), 300-307.

[25] A.B.Paalman-de Miranda, Topological semigroups, 2nd edition, (Mathematisch Centrum, Amsterdam, 1970).

[26] K.Numakura, Prime ideals and idempotents in compact semigroups, Duke Math. J. 24(1957), 671-680.

[27] J.S.Pym, Idempotent measures on semigroups, Pacific J. Math. 12(1962), 685-698.

[28] ________, Idempotent probability measures on compact semitopological semigroups, Proc. Amer. Math. Soc. 21(1969), 499-501.

[29] W.Rudin, Real and Complex analysis (McGraw-Hill, New York, 1966).

[30] J.G.Wendel, Haar measures and the semigroup of measures on a compact group, Proc. Amer. Math. Soc. 5(1954), 923-929.